

Well-Posedness of the Einstein–Euler System in Asymptotically Flat Spacetimes

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Abstract

We prove a local in time existence and uniqueness theorem of classical solutions of the coupled Einstein–Euler system, and therefore establish the well posedness of this system. We use the condition that the energy density might vanish or tends to zero at infinity and that the pressure is a certain function of the energy density, conditions which are used to describe simplified stellar models. In order to achieve our goals we are enforced, by the complexity of the problem, to deal with these equations in a new type of weighted Sobolev spaces of fractional order. Beside their construction, we develop tools for PDEs and techniques for elliptic and hyperbolic equations in these spaces. The well posedness is obtained in these spaces. The results obtained are related to and generalize earlier works of Rendall [35] for the Euler-Einstein system under the restriction of time symmetry and of Gamblin [17] for the simpler Euler–Poisson system.

1 Introduction

This paper deals with the Cauchy problem for the Einstein-Euler system describing a relativistic self-gravitating perfect fluid, whose density either has compact support or falls off at infinity in an appropriate manner, that is, the density belongs to a certain weighted Sobolev space.

The evolution of the gravitational field is described by the Einstein equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi T_{\alpha\beta} \quad (1.1)$$

where $g_{\alpha\beta}$ is a semi Riemannian metric having a signature $(-, +, +, +)$, $R_{\alpha\beta}$ is the Ricci curvature tensor, these are functions of $g_{\alpha\beta}$ and its first and second order partial derivatives

and R is the scalar curvature. The right hand side of (1.1) consists of the energy-momentum tensor of the matter, $T_{\alpha\beta}$ and in the case of a perfect fluid the latter takes the form

$$T^{\alpha\beta} = (\epsilon + p)u^\alpha u^\beta + pg^{\alpha\beta}, \quad (1.2)$$

where ϵ is the energy density, p is the pressure and u^α is the four-velocity vector. The vector u^α is a unit timelike vector, which means that it is required to satisfy the normalization condition

$$g_{\alpha\beta}u^\alpha u^\beta = -1. \quad (1.3)$$

The Euler equations describing the evolution of the fluid take the form

$$\nabla_\alpha T^{\alpha\beta} = 0, \quad (1.4)$$

where ∇ denotes the covariant derivative associated to the metric $g_{\alpha\beta}$. Equations (1.1) and (1.4) are not sufficient to determinate the structure uniquely, a functional relation between the pressure p and the energy density ϵ (equation of state) is also necessary. We choose an equation of state that has been used in astrophysical problems. It is the analogue of the well known polytropic equation of state in the non-relativistic theory, given by

$$p = f(\epsilon) = K\epsilon^\gamma, \quad K, \gamma \in \mathbb{R}^+, \quad 1 < \gamma. \quad (1.5)$$

The sound velocity is denoted by

$$\sigma^2 = \frac{\partial p}{\partial \epsilon}. \quad (1.6)$$

The unknowns of these equations are the semi Riemannian metric $g_{\alpha\beta}$, the velocity vector u^α and the energy density ϵ . These are functions of t and x^a where x^a ($a = 1, 2, 3$) are the Cartesian coordinates on \mathbb{R}^3 . The alternative notation $x^0 = t$ will also be used and Greek indices will take the values 0, 1, 2, 3 in the following.

The common method to solve the Cauchy problem for the Einstein equations consists usually of two steps. Unlike ordinary initial value problems, initial data must satisfy constraint equations intrinsic to the initial hypersurface. Therefore, the first step is to construct solutions of these constraints. The second step is to solve the evolution equations with these initial data, in the present case these are first order symmetric hyperbolic systems. As we describe later in detail, the complexity of our problems forces us to consider an additional third step, that is, after solving the constraint equations, we have to construct the initial data for the fluid equations.

The nature of this Einstein-Euler system (1.1), (1.4) and (1.5) forces us to treat both the constraint and the evolution equations in the same type of functional spaces. Under the above consideration, we have established the well posedness of this Einstein-Euler system in weighted Sobolev spaces of fractional order. Oliynyk has recently studied the Newtonian limit of this system in weighted Sobolev spaces of integer order [34]

We will briefly resume the situation in the mathematical theory of self gravitation perfect fluids describing compact bodies, such as stars: For the Euler-Poisson system Makino

proved a local existence theorem in the case the density has compact support and it vanishes at the boundary, [29]. Since the Euler equations are singular when the density ρ is zero, Makino had to regularize the system by introducing a new matter variable ($w = M(\rho)$). His solution however, has some disadvantages such as the fact they do not contain static solutions and moreover, the connection between the physical density and the new matter density remains obscure.

Rendall generalized Makino's result to the relativistic case of the Einstein–Euler equations, [35]. His result however suffers from the same disadvantages as Makino's result and moreover it has two essential restrictions: 1. Rendall assumed time symmetry, that means that the extrinsic curvature of the initial manifold is zero and therefore the Einstein's constraint equations are reduced to a single scalar equation; 2. Both the data and solutions are C_0^∞ functions. This regularity condition implies a severe restriction on the equation of state $p = K\epsilon^\gamma$, namely $\gamma \in \mathbb{N}$.

Similarly to Makino and Rendall, we have also used the Makino variable

$$w = M(\epsilon) = \epsilon^{\frac{\gamma-1}{2}}. \quad (1.7)$$

Our approach is motivated by the following observation. As it turns out, the system of evolution equations have the following form

$$A^0 \partial_t U + \sum_{k=1}^3 A^k \partial_k U = Q(\epsilon, ..), \quad (1.8)$$

where the unknown U consists of the gravitational field $g_{\alpha\beta}$ the velocity of the fluid u^α and the Makino variable w , and the lower order term Q contains the energy density ϵ . Thus, we need to estimate ϵ by w in the corresponding norm of the function spaces. Combining this estimation with the Makino variable (1.7), it results in an algebraic relation between the order of the functional space k and the coefficient γ of the equation of state (1.5) of the form

$$1 < \gamma \leq \frac{2+k}{k}. \quad (1.9)$$

This relation can be easily derived by considering $\|\partial^\alpha w\|_{L_2}$, $|\alpha| \leq k$. Moreover, it can be interpreted either as a restriction on γ or on k . Thus, unlike typical hyperbolic systems where often the regularity parameter is bounded from below, here we have both lower and upper bounds for differentiability conditions of the sort $\frac{5}{2} < k \leq \frac{2}{\gamma-1}$. A similar phenomenon for the Euler-Poisson equations was noted by Gamblin [17].

We want to interpret (1.9) as a restriction on k rather than on γ . Therefore, instead of imposing conditions on the equation of state and in order to sharpen the regularity conditions for existence theorems, we are lead to the conclusion of considering function spaces of fractional order, and in addition, the Einstein equations consist of quasi linear hyperbolic and elliptic equations. The only function spaces which are known to be useful for existence theorems of the constraint equations in the asymptotically flat case, are the

weighted Sobolev spaces $H_{k,\delta}$, $k \in \mathbb{N}$, $\delta \in \mathbb{R}$, which were introduced by Nirenberg and Walker, [33] and Cantor [6], and they are the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$(\|u\|_{k,\delta})^2 = \sum_{|\alpha| \leq k} \int ((1 + |x|)^{\delta+|\alpha|} |\partial^\alpha u|)^2 dx. \quad (1.10)$$

Hence we are forced to consider new function spaces $H_{s,\delta}$, $s \in \mathbb{R}$ which generalize the spaces $H_{k,\delta}$ to fractional order. The well posedness of the Einstein-Euler system is obtained in these spaces. In order to achieve this, we have to solve both the constraint and the evolution equations in the $H_{s,\delta}$ spaces.

Another difficulty which arises from the non-linear equation of state (1.5) is the compatibility problem of the initial data for the fluid and the gravitational field. There are three types of initial data for the Einstein-Euler system:

- The gravitational data is a triple (M, h, K) , where M is space-like manifold, $h = h_{ab}$ is a proper Riemannian metric on M and $K = K_{ab}$ is a second fundamental form on M (extrinsic curvature). The pair (h, K) must satisfy the constrain equations

$$\begin{cases} R(h) - K_{ab}K^{ab} + (h^{ab}K_{ab})^2 &= 16\pi z, \\ {}^{(3)}\nabla_b K^{ab} - {}^{(3)}\nabla^b (h^{bc}K_{bc}) &= -8\pi j^a, \end{cases} \quad (1.11)$$

where $R(h) = h^{ab}R_{ab}$ is the scalar curvature with respect to the metric h .

- The matter variables, consisting of the energy density z and the momentum density j^a , appear in the right hand side of the constraints (1.11).
- The initial data for Makino's variable w and the velocity vector u^α of the perfect fluid.

The projection of the velocity vector u^α , \bar{u}^α , on the tangent space of the initial manifold M leads to the following relations

$$\begin{cases} z &= \epsilon + (\epsilon + p)h_{ab}\bar{u}^a\bar{u}^b \\ j^\alpha &= (\epsilon + p)\bar{u}^a\sqrt{1 + h_{ab}\bar{u}^a\bar{u}^b} \end{cases} \quad (1.12)$$

between the matters variable (z, j^a) and (w, \bar{u}^a) . We cannot give w , \bar{u}^b and by relations (1.5) and (1.12) solve for z and j^α , since this is incompatible with the conformal scaling (see Section 4.1). In order to overcome this obstacle, we let $z = y^{\frac{2}{\gamma-1}}$ and $j^\alpha = v^\alpha/z$, then (1.12) is equivalent to (4.10) and the last one is invertible.

The paper is organized as follows: In the next section we perform the reduction of the Einstein-Euler system into a first order symmetric hyperbolic system. Choquet-Bruhat showed that the choice of harmonic coordinates converts the field equations (1.1) into wave equations which then can be written as a first order symmetric hyperbolic system

[10], [13], [19]. Reducing the Euler equations (1.4) to a first order symmetric hyperbolic system is not a trivial matter. We use a fluid decomposition and present a new reduction of the Euler equations. Beside having a very clear geometric interpretation, we give a complete description of the structure of the characteristics conformal cone of the system, namely, it is a union of a three-dimensional hyperplane tangent to the initial manifold and the sound cone.

In Section 3 we define the weighted Sobolev spaces of fractional order $H_{s,\delta}$ and present our main results. These include a solution of the compatibility problem, the construction of initial data and a solution to the evolution equations in the $H_{s,\delta}$ spaces. The announcement of the main results has been published in [5].

Section 4 deals with the constructions of the initial data. The common Lichnerowicz-York scaling method for solving the constraint equations cannot be applied here directly [13], [7], [42], since it violates the relations (1.12). We need to invert of (1.12) in order to construct the initial data and there are two conditions which guarantee it: the dominate energy condition $h_{ab}j^aj^b \leq z^2$, this is invariant under scaling; and the causality condition, this is the speed of sound (1.6) less than speed of light. Unfortunately the last condition is not invariant under scaling. It is also necessary to restrict the matter variables (z, j^a) to a certain region. We show the inversion of (1.12) exists provided that (z, j^a) belong to a certain region. This fact enables us to construct initial data for the evolution equations.

The local existence for first order symmetric hyperbolic systems in $H_{s,\delta}$ is discussed in Section 5. The known existence results in the H^s space [16], [24], [21], [38], [37], [28] cannot be applied to the $H_{s,\delta}$ spaces. The main difficulty here is the establishment of energy estimates for linear hyperbolic systems. In order to achieve it we have defined a specific inner-product in $H_{s,\delta}$ and in addition the Kato-Ponce commutator estimate [25], [38], [37] has an essential role in our approach. Once the energy estimates and other tools have been established in the $H_{s,\delta}$ space, we follow Majda's [28] iteration procedure and show existence, uniqueness and continuity in that norm.

In Section 6 we study elliptic theory in $H_{s,\delta}$ which is essential for the solution of the constraint equations. We will extend earlier results in weighted Sobolev spaces of integer order which were obtained by Cantor [7], Choquet-Bruhat and Christodoulou [11], Choquet-Bruhat, Isenberg and York [12], and Christodoulou and O'Murchadha [14] to the fractional ordered spaces. The central tool is a priori estimate for elliptic systems in the $H_{s,\delta}$ spaces (6.21). Its proof requires first the establishment of analogous a priori estimate in Bessel potential spaces H^s . Our approach is based on the techniques of pseudodifferential operators which have symbols with limited regularity and in order to achieve that we are adopting ideas being presented in Taylor's books [38] and [39]. A different method was derived recently by Maxwell [30] who also showed existence of solutions to Einstein constraint equations in vacuum in $H_{s,\delta}$ with the best possible regularity condition, namely $s > \frac{3}{2}$. The semi-linear elliptic equation is solved by following Cantor's homotopy argument [7] and generalized to the $H_{s,\delta}$ spaces.

Finally, in the Appendix we deal with of the construction, properties and tools for PDEs

in the weighted Sobolev spaces of fractional order $H_{s,\delta}$. Triebel extended the $H_{k,\delta}$ spaces given by the norm (1.10) to a fractional order [40], [41]. We present three equivalent norms, one of which is a combination of the norm (1.10) and the norm of Lipschitz-Sobolevskij spaces [36]. This definition is essential for the understanding of the relations between the integer and the fractional order spaces (see (7.3)). However the double integral makes it almost impossible to establish any property needed for PDEs. Throughout the effort to solve this problem, we were looking for an equivalent definition of the norm: we let $\{\psi_j\}_{j=0}^\infty$ be a dyadic resolution of unity in \mathbb{R}^3 and set

$$(\|u\|_{H_{s,\delta}})^2 = \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2j}\|_{H^s}^2, \quad (1.13)$$

where $(f)_\epsilon(x) = f(\epsilon x)$. When s is an integer, then the norms (1.10) and (1.13) are equivalent. Our guiding philosophy is to apply the known properties of the Bessel potential spaces H^s term-wise to each of the norms in the infinite sum (1.13) and in that way to extend them to the $H_{s,\delta}$ spaces. Of course, this requires a careful treatment and a sound consideration of the additional parameter δ . Among the properties which we have extended to the $H_{s,\delta}$ spaces are the algebra, Moser type estimates, fractional power, the embedding to the continuous and an intermediate estimate.

2 The Initial Value Problem for the Euler-Einstein System

This section deals with the reduction of the coupled evolution equations (1.1), (1.4) and (1.5) into a first order symmetric hyperbolic system.

2.1 The Euler equations written as a symmetric hyperbolic system

It is not obvious that the Euler equations written in the conservative form $\nabla_\alpha T^{\alpha\beta} = 0$ are symmetric hyperbolic. In fact these equations have to be transformed in order to be expressed in a symmetric hyperbolic form. Rendall presented such a transformation of these equations in [35], however, its geometrical meaning is not entirely clear and it might be difficult to generalize it to the non time symmetric case. Hence we will present a different hyperbolic reduction of the Euler equations and discuss it in some details, for we have not seen it anywhere in the literature. The basic idea is to perform the standard *fluid decomposition* and then to modify the equation by adding, in an appropriate manner, the normalization condition (1.3) which will be considered as a constraint equation.

The fluid decomposition method consists of:

1. The equation $\nabla_\nu T^{\nu\beta} = 0$ is once projected orthogonal onto u^α which leads to

$$u_\beta \nabla_\nu T^{\nu\beta} = 0. \quad (2.1)$$

2. The equation $\nabla_\nu T^{\nu\beta} = 0$ is projected into the rest space \mathcal{O} orthogonal to u^α of a fluid particle gives us:

$$P_{\alpha\beta} \nabla_\nu T^{\nu\beta} = 0 \quad \text{with} \quad P_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta, \quad P_{\alpha\beta} u^\beta = 0. \quad (2.2)$$

Inserting this decomposition into (1.2) results in a system in the following form:

$$u^\nu \nabla_\nu \epsilon + (\epsilon + p) \nabla_\nu u^\nu = 0; \quad (2.3a)$$

$$(\epsilon + p) P_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^\nu{}_\alpha \nabla_\nu p = 0. \quad (2.3b)$$

Note that we have beside the evolution equations (2.3a) and (2.3b) the following constraint equation: $g_{\alpha\beta} u^\alpha u^\beta = -1$. We will show later, in subsection 2.1.1 that this constraint equation is conserved under the evolution equation, that is, if it holds initially at $t = t_0$, then it will hold for $t > t_0$. Note that in most textbooks, the equation (2.3b) is presented as $(\epsilon + p) g_{\alpha\beta} u^\nu \nabla_\nu u^\beta + P^\nu{}_\alpha \nabla_\nu p = 0$, which is an equivalent form, since due to the normalization condition (1.3) we have $u_\beta \nabla_\nu u^\beta = 0$.

In order to obtain a symmetric hyperbolic system we have to modify it in the following way. The normalization condition (1.3) gives that $u_\beta u^\nu \nabla_\nu u^\beta = 0$, so we add $(\epsilon + p) u_\beta u^\nu \nabla_\nu u^\beta = 0$ to equation (2.3a) and $u_\alpha u_\beta u^\nu \nabla_\nu u^\beta = 0$ to (2.3b), which together with (1.6) results in,

$$u^\nu \nabla_\nu \epsilon + (\epsilon + p) P^\nu{}_\beta \nabla_\nu u^\beta = 0 \quad (2.4a)$$

$$\Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + \frac{\sigma^2}{(\epsilon + p)} P^\nu{}_\alpha \nabla_\nu \epsilon = 0, \quad (2.4b)$$

where $\Gamma_{\alpha\beta} = P_{\alpha\beta} + u_\alpha u_\beta = g_{\alpha\beta} + 2u_\alpha u_\beta$. As mentioned above we will introduce a new nonlinear matter variable which is given by (1.7). The idea which is behind this is the following: The system (2.4a) and (2.4b) is almost of symmetric hyperbolic form, it would be symmetric if we multiply the system by appropriate factors, for example, (2.4a) by $\frac{\partial p}{\partial \epsilon} = \sigma^2$ and (2.4b) by $(\epsilon + p)$. However, doing so we will be faced with a system in which the coefficients will either tend to zero or to infinity, as $\epsilon \rightarrow 0$. Hence, it is impossible to represent this system in a non-degenerate form using these multiplications.

The central point is now to introduce a new variable $w = M(\epsilon)$ which will regularize the equations even for $\epsilon = 0$. We do this by multiplying equation (2.4a) by $\kappa^2 M' = \kappa^2 \frac{\partial M}{\partial \epsilon}$. This results in the following system which we have written in matrix form:

$$\left(\begin{array}{c|c} \kappa^2 u^\nu & \kappa^2 (\epsilon + p) M' P^\nu{}_\beta \\ \hline \frac{\sigma^2}{(\epsilon + p) M'} P^\nu{}_\alpha & \Gamma_{\alpha\beta} u^\nu \end{array} \right) \nabla_\nu \begin{pmatrix} w \\ u^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (2.5)$$

In order to obtain symmetry we have to demand

$$M' = \frac{\sigma}{(\epsilon + p)\kappa}, \quad (2.6)$$

where $\kappa \gg 0$ has been introduced in order to simplify the expression for w . We choose κ so that

$$\frac{\sqrt{f'(\epsilon)}}{(\epsilon + p)\kappa} = \frac{2}{\gamma - 1} \frac{\epsilon^{\frac{\gamma-1}{2}}}{\epsilon}, \quad (2.7)$$

which gives the Makino variable (1.7). Taking into account the equation of state (1.5), we see that

$$\kappa = \frac{\gamma - 1}{2} \frac{\sqrt{K}\gamma}{1 + K\epsilon^{\gamma-1}} \gg 0. \quad (2.8)$$

Finally we have obtained the following system

$$\left(\begin{array}{c|c} \kappa^2 u^\nu & \sigma \kappa P^\nu{}_\beta \\ \hline \kappa \sigma P^\nu{}_\alpha & \Gamma_{\alpha\beta} u^\nu \end{array} \right) \nabla_\nu \begin{pmatrix} w \\ u^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (2.9)$$

which is both symmetric and non-degenerated. The covariant derivative ∇_ν takes in local coordinates the form $\nabla_\nu = \partial_\nu + \Gamma(g^{\gamma\delta}, \partial g_{\alpha\beta})$ which expresses the fact that the fluid u^α is coupled to equations (1.1) for the gravitational field $g_{\alpha\beta}$. In addition, from the definition of the Makino variable (1.7) we see that $\epsilon^{\gamma-1} = w^2$, so from the expression (1.6), it follows that $\sigma = \sqrt{\gamma K} w$ and κ —which is given by (2.8)—are C^∞ functions of w . Thus the fractional power of the equation of state (1.5) does not appear in the coefficients of the system (2.9), and these coefficients are C^∞ functions of the scalar w , the four vector u^α and the gravitational field $g_{\alpha\beta}$.

Let us now recall a general definition of symmetric hyperbolic systems.

Definition 2.1 (First order symmetric hyperbolic systems) *A quasilinear, symmetric hyperbolic system is a system of differential equations of the form*

$$L[U] = \sum_{\alpha=0}^4 A^\alpha(U; x) \partial_\alpha U + B(U; x) = 0 \quad (2.10)$$

where the matrices A^α are symmetric and for every arbitrary $U \in G$ there exists a covector ξ such that

$$\xi_\alpha A^\alpha(U; x) \quad (2.11)$$

is positive definite. The covectors ξ_α for which (2.11) is positive definite, are spacelike with respect to the equation (2.10). Both matrices A^α , B satisfy certain regularity conditions, which are going to be formulated later.

Usually ξ is chosen to be the vector $(1, 0, 0, 0)$ which implies via the condition (2.11) that the matrix A^0 has to be positive definite.

Now we want to show that A^0 of our system (2.9) is indeed positive definite. We do this in several steps.

1. Explicit computation of the principle symbol (2.9);
2. We show that $-u_\alpha$ is a space like covector with respect to the equations;
3. Then we apply a deformation argument and show that the covector $t_\alpha := (1, 0, 0, 0)$ is a space like covector with respect to the equation.

For each $\xi_\alpha \in T_x^*V$ the principle symbol is a linear map from $\mathbb{R} \times E_x$ to $\mathbb{R} \times F_x$, where E_x is a fiber in T_xV and F_x is a fiber in the cotangent space T_x^*V . Since in local coordinates $\nabla_\nu = \partial_\nu + \Gamma(g^{\gamma\delta}, \partial g_{\alpha\beta})$, the principle symbol of system (2.9) is

$$\xi_\nu A^\nu = \left(\begin{array}{c|c} \kappa^2(u^\nu \xi_\nu) & \sigma \kappa P^\nu{}_\beta \xi_\nu \\ \hline \sigma \kappa P^\nu{}_\alpha \xi_\nu & (u^\nu \xi_\nu) \Gamma_{\alpha\beta} \end{array} \right) \quad (2.12)$$

and the characteristics are the set of covectors for which $(\xi_\nu A^\nu)$ is not an isomorphism. Hence the characteristics are the zeros of $Q(\xi) := \det(\xi_\nu A^\nu)$.

The geometric advantages of the fluid decomposition are the following. The operators in the blocks of the matrix (2.12) are $P^\nu{}_\alpha$, the projection on the rest hyperplane \mathcal{O} and $\Gamma_{\alpha\beta}$, that is the reflection with respect to the same hyperplane. Therefore, the following relations hold:

$$\Gamma^{\alpha\gamma} \Gamma_{\gamma\beta} = \delta_\beta^\alpha, \quad \Gamma^{\alpha\gamma} P_\gamma{}^\nu = P^{\alpha\nu} \quad \text{and} \quad P_\beta{}^\alpha P_\alpha{}^\nu = P^\nu{}_\beta,$$

which yields

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Gamma^{\alpha\gamma} \end{array} \right) (\xi_\nu A^\nu) = \left(\begin{array}{c|c} \kappa^2(u^\nu \xi_\nu) & \sigma \kappa P^\nu{}_\beta \xi_\nu \\ \hline \sigma \kappa P^{\alpha\nu} \xi_\nu & (u^\nu \xi_\nu) (\delta_\beta^\alpha) \end{array} \right). \quad (2.13)$$

It is now fairly easy to calculate the determinate of the right hand side of (2.13) and we have

$$\det \left(\begin{array}{c|c} \kappa^2(u^\nu \xi_\nu) & \sigma \kappa P^\nu{}_\beta \xi_\nu \\ \hline \sigma \kappa P^{\alpha\nu} \xi_\nu & (u^\nu \xi_\nu) (\delta_\beta^\alpha) \end{array} \right) = \kappa^2(u^\nu \xi_\nu)^3 \left((u^\nu \xi_\nu)^2 - \sigma^2 P^{\alpha\nu} \xi_\nu P_\alpha{}^\nu \xi_\nu \right).$$

Since P_β^α is a projection,

$$P^{\alpha\nu}\xi_\nu P_\alpha^\nu\xi_\nu = g^{\nu\beta}\xi_\nu P_\beta^\alpha P_\alpha^\nu\xi_\nu = g^{\nu\beta}\xi_\nu P_\beta^\nu\xi_\nu = P_\beta^\nu\xi_\nu\xi^\beta \quad (2.14)$$

and since $\Gamma_\beta^\gamma : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is a reflection with respect to a hyperplane,

$$\det \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Gamma^{\alpha\gamma} \end{array} \right) = \det (g^{\alpha\beta}\Gamma_\beta^\gamma) = \det (g^{\alpha\beta}) \det (\Gamma_\beta^\gamma) = -(\det (g_{\alpha\beta}))^{-1}. \quad (2.15)$$

Consequently,

$$Q(\xi) := \det(\xi_\nu A^\nu) = -\kappa^2 \det(g_{\alpha\beta})(u^\nu \xi_\nu)^3 \{ (u^\nu \xi_\nu)^2 - \sigma^2 P_\beta^\alpha \xi_\alpha \xi^\beta \} \quad (2.16)$$

and therefore the characteristic covectors are given by two simple equations:

$$\xi_\nu u^\nu = 0; \quad (2.17)$$

$$(\xi_\nu u^\nu)^2 - \sigma^2 P_\beta^\alpha \xi_\alpha \xi^\beta = 0. \quad (2.18)$$

Remark 2.2 (*The structure of the characteristics conormal cone of*) *The characteristics conormal cone is therefore a union of two hypersurfaces in T_x^*V . One of these hypersurfaces is given by the condition (2.17) and it is a three dimensional hyperplane \mathcal{O} with the normal u^α . The other hypersurface is given by the condition (2.18) and forms a three dimensional cone the so called sound cone.*

Remark 2.3 *Equation (2.18) plays an essential role in determining whether the equations form a symmetric hyperbolic system.*

Let us now consider the timelike vector u_ν and the linear combination $-u_\nu A^\nu$, with A^ν from equation (2.9), we then obtain that

$$-u_\nu A^\nu = \left(\begin{array}{c|c} \kappa^2 & 0 \\ \hline 0 & \Gamma_{\alpha\beta} \end{array} \right) \quad (2.19)$$

is positive definite. Indeed, $\Gamma_{\alpha\beta}$ is a reflection with respect to a hyperplane. The normal of this hyperplane is a timelike vector. Hence, $-u_\nu$ is for the hydrodynamical equations a spacelike covector in the sense of partial differential equations. Herewith one has showed relatively elegant and elementary that the relativistic hydrodynamical equations are symmetric-hyperbolic.

Now we want however to show that the covector $t_\alpha = (1, 0, 0, 0)$ is spacelike with respect to the system (2.9). Since $P^\alpha_\beta u_\alpha = 0$, the covector $-u_\nu$ belongs to the sound cone

$$(\xi_\nu u^\nu)^2 - \sigma^2 P_\beta^\alpha \xi_\alpha \xi^\beta > 0. \quad (2.20)$$

Inserting $t_\nu = (1, 0, 0, 0)$ the right hand side of (2.20) yields

$$(u^0)^2(1 - \sigma^2) - \sigma^2 g^{00}. \quad (2.21)$$

Since the sound velocity is always less than the light speed, that is $\sigma^2 = \frac{\partial p}{\partial \epsilon} < c^2 = 1$, we conclude from (2.21) that t_ν also belongs to the sound cone (2.20). Hence, the vector $-u_\nu$ can be continuously deformed to t_ν while condition (2.20) holds along the deformation path. Consequently, the determinant of (2.16) remains positive under this process and hence $t_\nu A^\nu = A^0$ is also positive definite.

2.1.1 Conservation of the constraint equation $g_{\alpha\beta}u^\alpha u^\beta = -1$

Now it will be shown that the condition $g_{\alpha\beta}u^\alpha u^\beta = -1$, which acts as a constraint equation for the evolution equation, is conserved along stream lines u^α . Because, if for $t = t_0$ the condition $g_{\alpha\beta}u^\alpha u^\beta = -1$ holds and if it is conserved along stream lines, then $g_{\alpha\beta}u^\alpha u^\beta = -1$ holds also for $t > t_0$. So let $c(t)$ be a curve such that $c'(t) = u^\alpha$ and set $Z(t) = (u \circ c)_\beta (u \circ c)^\beta$, then we need to establish

$$\frac{d}{dt}Z(t) = 2u_\beta \nabla_{c'(t)} u^\beta = 2u^\nu u_\beta \nabla_\nu u^\beta = 0. \quad (2.22)$$

Multiplying the last four last rows of the Euler system (2.9) by u^α and recalling that P^ν_α is the projection on the rest space \mathcal{O} orthogonal to u^α , we have

$$\begin{aligned} 0 &= u^\alpha (\Gamma_{\alpha\beta} u^\nu \nabla_\nu u^\beta + \kappa \sigma P^\nu_\alpha \nabla_\nu w) \\ &= u^\alpha P_{\alpha\beta} u^\nu \nabla_\nu u^\beta - u^\nu u_\beta \nabla_\nu u^\beta + \kappa \sigma u^\alpha P^\nu_\alpha \nabla_\nu w \\ &= -u^\nu u_\beta \nabla_\nu u^\beta. \end{aligned}$$

2.2 The reduced Einstein field equations

In this paper we study the field equations (1.1) with the choice of the harmonic coordinate condition which takes the form

$$H^\alpha \equiv g^{\alpha\beta} g^{\gamma\delta} (\partial_\gamma g_{\beta\delta} - \frac{1}{2} \partial_\delta g_{\beta\gamma}) = 0. \quad (2.23)$$

When (2.23) is imposed, then the Einstein equations (1.1) convert to

$$g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} = H_{\alpha\beta}(g, \partial g) - 16\pi T_{\alpha\beta} + 8\pi g^{\mu\nu} T_{\mu\nu} g_{\alpha\beta}. \quad (2.24)$$

Hawking and Ellis proved the conservation of the harmonic coordinates for Einstein equations with matter including a perfect fluid [19]. Since (2.24) are quasi linear wave equations, the introducing auxiliary variables

$$h_{\alpha\beta\gamma} = \partial_\gamma g_{\alpha\beta}, \quad (2.25)$$

reduce them into a first order symmetric hyperbolic system:

$$\begin{aligned}
\partial_t g_{\alpha\beta} &= h_{\alpha\beta 0} \\
g^{ab} \partial_t h_{\gamma\delta a} &= g^{ab} \partial_a h_{\gamma\delta 0} \\
-g^{00} \partial_t h_{\gamma\delta 0} &= 2g^{0a} \partial_a h_{\gamma\delta 0} + g^{ab} \partial_a h_{\gamma\delta b} \\
&\quad + C_{\gamma\delta\alpha\beta\rho\sigma}^{\epsilon\zeta\eta\kappa\lambda\mu} h_{\epsilon\zeta\eta} h_{\kappa\lambda\mu} g^{\alpha\beta} g^{\rho\sigma} - 16\pi T_{\gamma\delta} + 8\pi g^{\rho\sigma} T_{\rho\sigma} g_{\gamma\delta}
\end{aligned} \tag{2.26}$$

The object $C_{\gamma\delta\alpha\beta\rho\sigma}^{\epsilon\zeta\eta\kappa\lambda\mu}$ is a combination of Kronecker deltas with integer coefficients. We therefore conclude:

Conclusion 2.4 (*The evolution equations in a first order symmetric hyperbolic form*) *The equations for Einstein gravitational field (1.1) coupled with the Euler equations (1.4) with the normalization conditions (1.3) and the equation of state (1.5), are equivalent to the system (2.26) and (2.9). The coupled systems (2.26) and (2.9) take the form of a first order symmetric hyperbolic system in accordance with Definition 2.1 and where A^0 is a positive definite matrix.*

3 New Function Spaces and the Principle Results

Our principle results concern the solution of the compatibility of the initial data for the equations of the fluid and the gravitational field (1.12), solution to Einstein constraint equations (1.11) and solution to the coupled evolution equations (1.1) and (1.4).

The coupled evolution equations (1.1) and (1.4) are equivalent to the first order symmetric hyperbolic systems (2.9) and (2.26) as we have shown in section 2.1. The initial data for these coupled systems cannot be given freely, therefore they are constructed in the following way. Firstly the compatibility of the initial data for the fluid and the gravitational field (1.12) have to be solved and next the constraint equations (1.11), which lead to an elliptic system. The presence of an equation of state (1.5) compels us to treat both the elliptic and hyperbolic equations in the weighted Sobolev spaces of fractional order.

The Bessel potential spaces H^s which are the natural choice for the hyperbolic systems are inappropriate for the solutions of the constraint equations in asymptotically flat manifolds. Roughly speaking, because the Laplacian is not invertible in these spaces.

As we explained in the introduction, the Nirenberg-Walker-Cantor weighted Sobolev spaces of integer order $H_{m,\delta}$ [6], [33] are suitable for the solutions of the constraints in asymptotically flat manifolds. Their norm is given by (1.10).

We first define the *weighted fractional Sobolev spaces*. We make a dyadic resolution of the unity in \mathbb{R}^3 as follows. Let $K_j = \{x : 2^{j-3} \leq |x| \leq 2^{j+2}\}$, ($j = 1, 2, \dots$) and $K_0 = \{x : |x| \leq 4\}$. Let $\{\psi_j\}_{j=0}^\infty$ be a sequence of $C_0^\infty(\mathbb{R}^3)$ such that $\psi_j(x) = 1$ on K_j , $\text{supp}(\psi_j) \subset \cup_{l=j-4}^{j+3} K_l$, for $j \geq 1$ and $\text{supp}(\psi_0) \subset K_0 \cup K_1$.

We denote by H^s the Bessel potential spaces with the norm ($p = 2$)

$$\|u\|_{H^s}^2 = c \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi,$$

where \hat{u} is the Fourier transform of u . Also, for a function f , $f_\varepsilon(x) = f(\varepsilon x)$.

Definition 3.1 (Weighted fractional Sobolev spaces: infinite sum of semi norms) For $s \geq 0$ and $-\infty < \delta < \infty$,

$$(\|u\|_{H_{s,\delta}})^2 = \sum_j 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{(2^j)}\|_{H^s}^2. \quad (3.1)$$

The space $H_{s,\delta}$ is the set of all temperate distributions with a finite norm given by (3.1).

3.1 The principle results

3.1.1 The compatibility of the initial data for the fluid and the gravitational field

The matter data (non-gravitational) (z, j) which appear in the right hand side of (1.11) are coupled to the initial data of the perfect fluid (1.2) via the relations (1.12). Thus, an indispensable condition for obtaining a solution of the Einstein-Euler system is the inversion of (1.12). This system is not invertible for all $(z, j^a) \in \mathbb{R}_+ \times \mathbb{R}^3$, but the inverse does exist in a certain region.

Theorem 3.2 (Reconstruction theorem for the initial data) There is a real function $S : [0, 1) \rightarrow \mathbb{R}$ such that if

$$0 \leq z < S(\sqrt{h_{ab}j^aj^b}/z), \quad (3.2)$$

then system (1.12) has a unique inverse. Moreover, the inverse mapping is continuous in $H_{s,\delta}$ norm.

Remark 3.3 The matter initial data (z, j^a) for the Einstein-Euler system with the equation of state (1.5) cannot be given freely. They must satisfy condition (3.2). This condition includes the inequality

$$z^2 \geq h_{ab}j^aj^b, \quad (3.3)$$

which is known as the dominate energy condition.

3.1.2 Solution to the constraint equations

The gravitational data is a triple (M, h, K) , where M is a space-like asymptotically flat manifold, $h = h_{ab}$ is a proper Riemannian metric on M , and $K = K_{ab}$ is the second fundamental form on M (extrinsic curvature). The metric h_{ab} and the extrinsic curvature K must satisfy Einstein's constraint equations (1.11). The free initial data is a set $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^a)$, where \bar{h}_{ab} is a Riemannian metric, \bar{A}_{ab} is divergence and trace free form, \hat{y} is a scalar function and \hat{v}^a is a vector.

Theorem 3.4 (Solution of the constraint equations) *Given free data $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^a)$ such that $(\bar{h}_{ab} - I) \in H_{s,\delta}$, $\bar{A}_{ab} \in H_{s-1,\delta+1}$, $(\hat{y}, \hat{v}^a) \in H_{s-1,\delta+2}$, $\frac{5}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$.*

(i) *Then there exists two positive functions α and ϕ such that $(\alpha - 1), (\phi - 1) \in H_{s,\delta}$, a vector field $W \in H_{s,\delta}$ such that the gravitational data*

$$h_{ab} = (\phi\alpha)^4 \bar{h}_{ab} \quad \text{and} \quad K_{ab} = (\phi\alpha)^{-2} \bar{A}_{ab} + \phi^{-2} \hat{\mathcal{L}}(W) \quad (3.4)$$

satisfy the constraint equations (1.11) with $z = \phi^{-8} \hat{y}^{\frac{2}{\gamma-1}}$ and $j^b = \phi^{-10} \hat{y}^{\frac{2}{\gamma-1}} \hat{v}^b$ as the right hand side, here $\hat{\mathcal{L}}$ is the Killing vector field operator. In addition, the $H_{s,\delta} \times H_{s-1,\delta+1}$ norms of $(h_{ab} - I, K_{ab})$ depend continuously on the $H_{s,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$ norms of $(\bar{h}_{ab} - I, \bar{A}_{ab}, \hat{y}, \hat{v}^a)$.

(ii) *Let $\hat{h}_{ab} = \alpha^4 \bar{h}_{ab}$, $\hat{z} = \hat{y}^{\frac{2}{\gamma-1}}$, $j^a = \hat{y}^{\frac{2}{\gamma-1}} \hat{v}^a$ and Ω^{-1} denote the inverse of relations (1.12). If $(\hat{h}_{ab}, \hat{z}, \hat{j}^a)$ satisfies (3.2), then the data for the four velocity vector and Makino variable are given by: $z = \phi^{-8} \hat{z}$, $j^a = \phi^{-10} \hat{j}^a$,*

$$(w, \bar{u}^a) := \Omega^{-1}(z, j^a) \quad \text{and} \quad \bar{u}^0 = 1 + h_{ab} \bar{u}^a \bar{u}^b \quad (3.5)$$

and they satisfy the compatibility conditions (1.12). In addition, the $H_{s-1,\delta+2}$ norms of $(w, \bar{u}^a, u^0 - 1)$ depend continuously on the $H_{s,\delta} \times H_{s-1,\delta+2}$ norms of $(\bar{h}_{ab} - I, \hat{y}, \hat{j}^a)$.

3.1.3 Solution to the evolution equations

The unknowns of the evolution equations are the gravitational field $g_{\alpha\beta}$ and its first order partial derivatives $\partial_\alpha g_{\gamma\delta}$, the Makino variable w and the velocity vector u^α . We represent them by the vector $U = (g_{\alpha\beta} - \eta_{\alpha\beta}, \partial_\alpha g_{\gamma\delta}, \partial_0 g_{\gamma\delta}, w, u^\alpha, u^0 - 1)$, here $\eta_{\alpha\beta}$ denotes the Minkowski metric. The initial data for equation (2.26) are

$$g_{ab}|_M = h_{ab}, \quad g_{0b}|_M = 0, \quad g_{00}|_M = -1, \quad -\frac{1}{2} \partial_0 g_{ab}|_M = K_{ab}, \quad (3.6)$$

where (h_{ab}, K_{ab}) are given by (3.4), and (3.5) for equation (2.9). Theorem 3.4 guarantees that they satisfy the constraints (1.11) and the compatibility condition (1.12).

Theorem 3.5 (*Solutions of the evolution equations (2.26) and (2.9)*) *Let $\frac{7}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Given the solutions of the constraint equations as described in Theorem 3.4, then there exists a $T > 0$, a unique semi-Riemannian metric $g_{\alpha\beta}$ solution to (2.26) and a unique pair (w, u^α) solution to (2.9)*

$$(g_{\alpha\beta} - \eta_{\alpha\beta}) \in C([0, T], H_{s,\delta}) \cap C^1([0, T], H_{s-1,\delta+1}) \quad (3.7)$$

$$(w, u^\alpha, u^0 - 1) \in C([0, T], H_{s-1,\delta+2}) \cap C^1([0, T], H_{s-2,\delta+3}). \quad (3.8)$$

4 The Initial Data

The Cauchy problem for Einstein fields equations (1.1) coupled with the Euler system (1.4) consists of solving coupled hyperbolic systems (2.26) and (2.9) with given initial data. There are two types of data for equations (1.1), gravitational and matter data.

The gravitational data is a triple (M, h, K) , where M is a space-like manifold, $h = h_{ab}$ is a proper Riemannian metric on M and $K = K_{ab}$ is the second fundamental form on M (extrinsic curvature).

Let n be the unit normal to the hypersurface M , $\delta_\beta^\alpha + n^\alpha n_\beta$ be the projection on M and define

$$z = T_{\alpha\beta} n^\alpha n^\beta, \quad (4.1)$$

$$j^\alpha = (\delta_\gamma^\alpha + n^\alpha n_\gamma) T^{\gamma\beta} n_\beta. \quad (4.2)$$

The scalar z is the energy density and the vector j^α is the momentum density. These quantities are called matter variables and they appear as sources in the constraint equations (4.7) and (4.8) below.

In conjunction with these we must supply initial data for the velocity vector u^α . So we apply the projection to u^α and set $\bar{u}^\alpha = (\delta_\beta^\alpha + n^\alpha n_\beta) u^\beta$. Then from the relation of the perfect fluid (1.2), (4.1), and (4.2) we see that

$$z = (\epsilon + p)(n_\beta u^\beta)^2 - p, \quad (4.3)$$

$$j^\alpha = (\epsilon + p)\bar{u}^\alpha(n_\beta u^\beta). \quad (4.4)$$

The vectors j^α and \bar{u}^α are tangent to the initial surface and so they can be identified with vectors j^a and \bar{u}^a intrinsic to this surface. Recalling the normalization condition (1.3), we have $-1 = -(n_\beta u^\beta)^2 + h_{ab}\bar{u}^a\bar{u}^b$. Thus the matter data (z, j^a) can be identified with the initial data for the velocity vector as follow:

$$z = \epsilon + (\epsilon + p)h_{ab}\bar{u}^a\bar{u}^b, \quad (4.5)$$

$$j^\alpha = (\epsilon + p)\bar{u}^a\sqrt{1 + h_{ab}\bar{u}^a\bar{u}^b}. \quad (4.6)$$

These two types of data cannot be given freely, because the hypersurface (M, h) is a sub-manifold of (V, g) , therefore the Gauss Codazzi equations lead to Einstein constraint equations

$$R(h) - K_{ab}K^{ab} + (h^{ab}K_{ab})^2 = 16\pi z \quad (\text{Hamiltonian constraint}) \quad (4.7)$$

$${}^{(3)}\nabla_b K^{ab} - {}^{(3)}\nabla^b (h^{bc}K_{bc}) = -8\pi j^a \quad (\text{momentum constraint}). \quad (4.8)$$

Here $R(h) = h^{ab}R_{ab}$ is the scalar curvature with respect to the metric h .

We turn now to the conformal method which allows us to construct the solutions of the constraint equations (4.7) and (4.8). Before entering into details we have to discuss the relations between the initial data for the system of Einstein gravitational fields (1.1) and Euler equations (1.4) which are given by (4.5) and (4.6). As it turns out this relations is by no means trivial, and indeed they will force us to modify the conformal method.

4.1 The compatibility problem of the initial data for the fluid and the gravitational fields

On the one hand, the initial data for the Euler equations are $w(\epsilon)$ and u^α . On the other hand $z = F(w(\epsilon), \bar{u}^a)$ and $j^a = H(w(\epsilon), \bar{u}^a)$, which are given by (4.5) and (4.6) respectively, appear as sources in the constraint equations (4.7) and (4.8). There we have the possibility of either to consider w and u^α as the fundamental quantities and construct then z and j^a or, vice verse, to consider z and j^a as the fundamental quantities and construct then w and u^α .

The first possibility does not work because the geometric quantities which occur on the left hand side of the constraint equations are supposed to scale with some power of a scalar function ϕ . So z and j^a , which are the sources in the constraint equations, must also scale with a definite power of ϕ . If ϵ is scaled with a certain power of ϕ , then p would be scaled, according to the equation of state (1.5), to a different power. Hence, by (4.5) z is a sum of different powers. Thus, the power which ϵ and p are scaled would have to be zero and they would be left unchanged by the rescaling. Similarly it can be seen that \bar{u}^a would remain unchanged. So in fact z would be unchanged and this is inconsistent with the scalding used in the conformal method.

Instead of constructing (w, \bar{u}^a) from (z, j^a) it is more useful to introduce some auxiliary quantities. Beside the Makino variable $w = \epsilon^{\frac{\gamma-1}{2}}$, we set

$$y = z^{\frac{\gamma-1}{2}} \quad \text{and} \quad v^a = \frac{j^a}{z}. \quad (4.9)$$

Now we consider the following map

$$\Phi \begin{pmatrix} w \\ \bar{u}^a \end{pmatrix} = \begin{pmatrix} w[1 + (1 + Kw^2)h_{ab}\bar{u}^a\bar{u}^b]^{\frac{\gamma-1}{2}} \\ \frac{(1+Kw^2)\bar{u}^a\sqrt{1+h_{bc}\bar{u}^b\bar{u}^c}}{1+(1+Kw^2)h_{bc}\bar{u}^b\bar{u}^c} \end{pmatrix} = \begin{pmatrix} y \\ v^a \end{pmatrix}, \quad (4.10)$$

which is equivalent to equations (4.5) and (4.6). The initial data (w, \bar{u}^a) for the fluid are reconstructed through the inversion of Φ above.

Theorem 4.1 (Reconstruction theorem for the initial data) *There is a function $s : [0, 1) \rightarrow \mathbb{R}$ such that the map Φ defined by (4.10) is a diffeomorphism from $[0, (\sqrt{\gamma K})^{-\frac{1}{2}}) \times \mathbb{R}^3$ to Ω , where*

$$\Omega = \{(y, v^a) : 0 \leq y < s(\sqrt{h_{ab}v^av^b}), h_{ab}v^av^b < 1\}. \quad (4.11)$$

Proof (of theorem 4.1) Let $\rho = \sqrt{h_{ab}\bar{u}^a\bar{u}^b}$, \bar{u}_0 be a unit vector and $R_{\bar{u}^a}$ be the rotation with respect to the metric h_{ab} such that $\bar{u}^a = \rho R_{\bar{u}^a}\bar{u}_0$. Then

$$\Phi \begin{pmatrix} w \\ \bar{u}^a \end{pmatrix} = \Phi \begin{pmatrix} w \\ \rho R_{\bar{u}^a}\bar{u}_0 \end{pmatrix} = \begin{pmatrix} w[1 + (1 + Kw^2)\rho^2]^{\frac{\gamma-1}{2}} \\ \frac{(1+Kw^2)R_{\bar{u}^a}\bar{u}_0\rho\sqrt{1+\rho^2}}{1+(1+Kw^2)\rho^2} \end{pmatrix}. \quad (4.12)$$

Therefore, we can first invert the two dimensional map

$$\Theta \begin{pmatrix} w \\ \rho \end{pmatrix} := \begin{pmatrix} w[1 + (1 + Kw^2)\rho^2]^{\frac{\gamma-1}{2}} \\ \frac{(1+Kw^2)\rho\sqrt{1+\rho^2}}{1+(1+Kw^2)\rho^2} \end{pmatrix} \quad (4.13)$$

for $\rho \geq 0$ and then apply again the rotation. For $w > 0$, we decompose Θ of (4.13) as follows:

$$\begin{pmatrix} w \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \epsilon \\ \rho \end{pmatrix} \mapsto \begin{pmatrix} \epsilon + (\epsilon + p(\epsilon))\rho^2 \\ (\epsilon + p(\epsilon))\rho\sqrt{1+\rho^2} \end{pmatrix} =: \begin{pmatrix} z \\ r \end{pmatrix} \mapsto \begin{pmatrix} z^{\frac{\gamma-1}{2}} \\ \frac{r}{z} \end{pmatrix}. \quad (4.14)$$

In order to show that this is a one to one map, we need to show that the Jacobian of $G(\epsilon, \rho) := (\epsilon + (\epsilon + p(\epsilon))\rho^2, (\epsilon + p(\epsilon))\rho\sqrt{1+\rho^2})$ does not vanish. This computation results with

$$\det \begin{pmatrix} 1 + (1 + p')\rho^2 & (1 + p')\rho\sqrt{1+\rho^2} \\ (\epsilon + p)2\rho & (\epsilon + p)\frac{1+2\rho^2}{\sqrt{1+\rho^2}} \end{pmatrix} = \frac{(\epsilon + p)}{\sqrt{1+\rho^2}} (1 + \rho^2(1 - p')). \quad (4.15)$$

Recall that the speed of sound is given by (1.6), therefore the causality condition $\sigma^2 < c^2 = 1$ imposes the below restriction of the domain of definition of the map Θ :

$$\sigma^2 = p' = \frac{\partial p}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} (K\epsilon^\gamma) = \gamma K\epsilon^{\gamma-1} = \gamma Kw^2 < 1. \quad (4.16)$$

Let S be the strip $\{0 \leq w < (\sqrt{\gamma K})^{-\frac{1}{2}}, 0 \leq \rho < \infty\}$. We now want to show that $\Theta : S \rightarrow \Theta(S)$ is a bijection. Clearly, $\Theta(0, \rho) = (0, \frac{\rho}{\sqrt{1+\rho^2}})$ maps $\{0\} \times [0, \infty)$ to $\{0\} \times [0, 1)$

in a one to one manner, and $\Theta(w, 0) = (w, 0)$ is of course a bijection. The line $(\sqrt{\gamma K})^{-\frac{1}{2}}, \rho)$ is mapped to the curve

$$\begin{pmatrix} y(\rho) \\ x(\rho) \end{pmatrix} = \begin{pmatrix} (\sqrt{\gamma K})^{-\frac{1}{2}} (1 + 2\rho^2)^{\frac{\gamma-1}{2}} \\ \frac{2\rho\sqrt{1+\rho^2}}{1+2\rho^2} \end{pmatrix}. \quad (4.17)$$

Since $\frac{dx}{d\rho} > 0$, there exists a function $s : [0, 1) \rightarrow \mathbb{R}$ such that the curve (4.17) is given by the graph of s and the image of Θ is the set below this graph, that is,

$$\Theta(S) = \{(y, x) : y < s(x), 0 \leq x < 1\}. \quad (4.18)$$

By (4.14), (4.15) and (4.16) we conclude that the Jacobian of the map Θ does not vanish in the interior of S , hence $\Theta : S \rightarrow \Theta(S)$ is locally one to one map. It is well known that a locally one to one map between two simply connected sets is a bijective map. ■

4.2 Cantor's conformal method for solving the constraint equations

In principle there are two possibilities for solving the constraint equation for an asymptotically flat manifold:

- Either to adapt directly the method of York et al, but then one is forced to impose certain relations between $R(\bar{h})$ and the second fundamental form (see Choquet-Bruhat and York [13] for details).
- These undesirable conditions can be substituted by a method developed by Cantor and we will describe it in the following. (This method has been discussed in detail in the literature, see for example [2], [13], [7] [14] and reference therein.)

In this method parts of the data are chosen (the so-called free data), and the remaining parts are determined by the constraint equations (4.7) and (4.8). The free initial data are $(\bar{h}_{ab}, \bar{A}_{ab}, \bar{z}, \bar{j})$, where A_{ab} is a divergence and trace free 2-tensor. The main idea is to consider two conformal scaling functions, α and ϕ .

1. We start with $\hat{h}_{ab} = \alpha^4 \bar{h}_{ab}$. If α is a positive solution to (4.24), then $R(\hat{h}) = 0$. The Brill-Cantor condition (see Definition 4.5) is necessary and sufficient for the existence of positive solutions. Having solved equation (4.24), we now adjust the given data to the new metric: $\hat{A}^{ab} = \alpha^{-10} \bar{A}^{ab}$, $\hat{z} = \alpha^{-8} \bar{z}$ and $\hat{j}^a = \alpha^{-10} \bar{j}^a$.
2. The second step here is solve the Lichnerowicz Laplacian (4.28) and set

$$\hat{K}^{ab} = (\mathcal{L}(W))^{ab} + \alpha^{-10} A^{ab}, \quad (4.19)$$

where $(\mathcal{L}(W))^{ab}$ is the Killing operator giving by (4.26).

3. The third step is: If ϕ is a solution to the Lichnerowicz equation (4.29), then it follows from (4.30) that the data $h_{ab} = \phi^4 \hat{h}_{ab}$, $K^{ab} = \phi^{-10} \hat{K}^{ab}$, $z = \phi^{-8} \hat{z}$ and $j^a = \phi^{-10} \hat{j}^a$ satisfy the constraint equations (4.7) and (4.8).

For the Einstein-Euler system with the equation of state (1.5) it is essential that the initial data will satisfy condition (4.11) of Theorem 4.1. Therefore in this case it is necessary to adjust this method.

Here the free initial data are:

$$(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^b). \quad (4.20)$$

where \bar{A}_{ab} is trace and divergence free, that is, $\bar{D}_a \bar{A}^{ab} = \bar{h}_{ab} \bar{A}^{ab} = 0$, where \bar{D}_a is the covariant derivative with respect to the metric \bar{h}_{ab} . We require that the matter data (\hat{y}, \hat{v}^a) , will satisfy the condition

$$0 \leq \hat{y} < s \left(\sqrt{\hat{h}_{ab} \hat{v}^a \hat{v}^b} \right), \quad (4.21)$$

where $s(\cdot)$ is given by (4.18). The remaining initial data are determined by the constraint equations (4.7) and (4.8), relations (4.9) and Theorem 4.1.

Remark 4.2 *The distinction between the gravitational data $(\bar{h}_{ab}, \bar{A}_{ab})$ and the matter data (\hat{z}, \hat{j}^b) is caused by condition (4.11). For if we make the scaling $\hat{h}_{ab} = \phi^4 \bar{h}_{ab}$, $\hat{z} = \phi^{-8} \bar{z}$, and $\hat{j}^b = \phi^{-10} \bar{j}^b$, then $\hat{v}^b = \phi^{-2} \bar{v}^b$, $\hat{y} = \phi^{-4(\gamma-1)} \bar{y}$ and $\hat{h}_{ab} \hat{v}^a \hat{v}^b = \bar{h}_{ab} \bar{v}^a \bar{v}^b$. Thus under this conformal transformation, the argument of s in (4.11) is invariant, while the left hand side will be affected. Therefore the free initial data are partially invariant under conformal transformations.*

Now, if we perform the conformal transformation

$$\hat{h}_{ab} = \alpha^4 \bar{h}_{ab}, \quad (4.22)$$

then the scalar curvature with respect to the metric \hat{h}_{ab} , $R(\hat{h})$, satisfies

$$-8\Delta_{\bar{h}}\alpha + R(\bar{h})\alpha = R(\hat{h})\alpha^5. \quad (4.23)$$

Therefore, if there exists a nonnegative solution to the equation

$$-\Delta_{\bar{h}}\alpha + \frac{1}{8}R(\bar{h})\alpha = 0, \quad (4.24)$$

then the metric \hat{h}_{ab} given by (4.22) will have zero scalar curvature. We continue the construction as follow. Let $\hat{A}^{ab} = \alpha^{-10} \bar{A}^{ab}$, \hat{D}_a denotes the covariant derivative with respect to the metric \hat{h}_{ab} , since $\hat{D}_a \hat{A}^{ab} = \alpha^{-10} \bar{D}_a \bar{A}^{ab}$, \hat{A}^{ab} is a divergence and trace free 2 tensor.

Assume \hat{K} is a symmetric covariant 2-tensor which satisfies the maximal slice condition, that is $\hat{h}_{ab}\hat{K}^{ab} = 0$. Then we split \hat{K} by writing it for some vector W :

$$\hat{K}^{ab} = \hat{A}^{ab} + \hat{\mathcal{L}}^{ab}(W), \quad (4.25)$$

where $\hat{\mathcal{L}}$ is the Killing field operator

$$\left(\hat{\mathcal{L}}(W)\right)^{ab} = \left(\hat{\mathcal{L}}_W \hat{h}\right)^{ab} - \frac{1}{3}\hat{h}^{ab}\text{Tr}\hat{\mathcal{L}}_W \hat{h} = \hat{D}_a W^b + \hat{D}_b W^a - \frac{1}{3}\hat{h}^{ab}\text{Tr}\hat{\mathcal{L}}_W \hat{h}, \quad (4.26)$$

and $\hat{\mathcal{L}}_W \hat{h}$ is the Lie derivative. The momentum constraint (4.8) is now equivalent to

$$\hat{D}_a \hat{K}^{ab} = \hat{D}_a \left(\hat{\mathcal{L}}(W)\right)^{ab} = -8\pi \hat{j}^b, \quad (4.27)$$

that is, W is a solution to the Lichnerowicz Laplacian system

$$\left(\Delta_{L_{\hat{h}}} W\right)^b := \hat{D}_a \left(\hat{\mathcal{L}}(W)\right)^{ab} = \Delta_{\hat{h}} W + \frac{1}{3}\hat{D}^b \left(\hat{D}_a W^a\right) + \hat{R}_a^b W^a = -8\pi \hat{j}^b, \quad (4.28)$$

here \hat{R}_a^b is the Ricci curvature tensor with respect to the metric \hat{h}_{ab} .

Having solved the Lichnerowicz Laplacian (4.28), we consider the Lichnerowicz equation

$$-\Delta_{\hat{h}} \phi = 2\pi \hat{z} \phi^{-3} + \frac{1}{8} \hat{K}_a^b \hat{K}_b^a \phi^{-7}. \quad (4.29)$$

Now we put $h_{ab} = \phi^4 \hat{h}_{ab}$, $K_{ab} = \phi^{-2} \hat{K}_{ab}$, $z = \phi^{-8} \hat{z}$ and $j^b = \phi^{-10} \hat{j}^b$. Since

$$-\Delta_{\hat{h}} \phi = \phi^5 \frac{1}{8} R(h) = \phi^5 \left(2\pi z + \frac{1}{8} K_a^b K_b^a\right) = \phi^5 \left(2\pi \hat{z} \phi^{-8} + \frac{1}{8} \hat{K}_a^b \hat{K}_b^a \phi^{-12}\right), \quad (4.30)$$

the Hamiltonian constraint (4.7) is valid, and since

$$D_a K^{ab} = \phi^{-10} \hat{D}_a \hat{K}^{ab} = -8\pi \phi^{-10} \hat{j}^b = -8\pi j^b \quad (4.31)$$

the data (h_{ab}, K_{ab}, z, j^b) satisfy the constraint equations (4.7) and (4.8). In order that the matter variables and (z, j) satisfy the compatibility conditions (4.5) and (4.6) it is necessary to check that $y = z^{\frac{\gamma-1}{2}} = (\phi^{-8} \hat{z})^{\frac{\gamma-1}{2}} = \phi^{-4(\gamma-1)} \hat{y}$ and $v^b = \frac{j^b}{z} = \phi^{-2} \hat{v}^b$ satisfy condition (4.11). Indeed,

$$0 \leq y < s \left(\sqrt{h_{ab} v^a v^b} \right) \Leftrightarrow 0 \leq \phi^{-4(\gamma-1)} \hat{y} < s \left(\sqrt{\hat{h}_{ab} \hat{v}^a \hat{v}^b} \right), \quad (4.32)$$

but since $\phi \geq 1$, $\phi^{-4(\gamma-1)} \hat{y} \leq \hat{y}$ and thus assumption (4.21) assures condition (4.11).

Theorem 4.3 (Construction of the gravitational data) *Given the free data $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^b)$ such that $(\bar{h}_{ab} - I) \in H_{s,\delta}$, $\bar{A}_{ab} \in H_{s-1,\delta+1}$, $(\hat{y}, \hat{v}^b) \in H_{s-1,\delta+2}$, $\frac{5}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Then the gravitational data:*

$$h_{ab} = (\phi\alpha)^4 \bar{h}_{ab} \quad \text{and} \quad K_{ab} = (\phi\alpha)^{-2} \bar{A}_{ab} + \phi^{-2} \hat{\mathcal{L}}(W)$$

satisfy the constraint equations (4.7) and (4.8) with $z = \phi^{-8} \hat{y}^{\frac{2}{\gamma-1}}$ and $j^b = \phi^{-10} \hat{y}^{\frac{2}{\gamma-1}} \hat{v}^b$ as the right hand side. In addition, $(h_{ab} - I) \in H_{s,\delta}$ and $K_{ab} \in H_{s-1,\delta+1}$ and therefore if $\frac{7}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$, then these data have the needed regularity so they can serve as initial data for the evolution equations of the reduced Einstein equations (2.26).

Proof (of Theorem 4.3)

- The free data are $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^b)$, where $(\bar{h}_{ab} - I) \in H_{s,\delta}$, $\bar{A}_{ab} \in H_{s-1,\delta+1}$ a divergence a trace free 2-tensor and $(\hat{y}, \hat{v}^b) \in H_{s-1,\delta+2}$.
- The function α satisfies equation (4.24), so by Theorem 4.6 $\alpha > 0$ and $(\alpha - 1) \in H_{s,\delta}$ provided that $s \geq 2$ and $\delta > -\frac{3}{2}$. Since α is continuous and $\lim_{|x| \rightarrow \infty} \alpha(x) - 1 = 0$, there is a compact set D of \mathbb{R}^3 such that $\alpha(x) \geq \frac{1}{2}$ for $x \notin D$ and $\min_D \alpha(x) \geq t_0 > 0$.
- The function $F(t) := \frac{1-t}{t}$ has bounded derivatives in $[\min\{t_0, \frac{1}{2}\}, \infty)$, so by Moser type estimate, Theorem 8.11, $\alpha^{-1} - 1 = \frac{1-\alpha}{\alpha} \in H_{s,\delta}$.
- Now, by algebra (Proposition 8.8), $(\hat{h}_{ab} - I) = (\alpha^4 \bar{h}_{ab} - I) \in H_{s,\delta}$ and $\hat{A}^{ab} = \alpha^{-10} \bar{A}^{ab} \in H_{s-1,\delta+1}$.
- The matter variables (\hat{z}, \hat{j}^b) are given by $\hat{z} = \hat{y}^{\frac{2}{\gamma-1}}$, $\hat{j}^b = \hat{z} \hat{v}^b$. Therefore Proposition 8.9 implies that $\hat{z} \in H_{s-1,\delta+2}$, provided that $\frac{3}{2} < s-1 < \frac{2}{\gamma-1} + \frac{1}{2}$ and also $j^b \in H_{s-1,\delta+2}$ by the Proposition 8.8.
- The vector W is a solution of the Lichnerowicz Laplacian (4.28), thus according to Theorem 4.8 below, $W \in H_{s,\delta}$ if $s \geq 2$. Hence \hat{K}^{ab} given in (4.25) belongs to $H_{s-1,\delta+1}$. Again, by Proposition 8.8, $\hat{K}_a^b \hat{K}_b^a \in H_{s-2,\delta+2}$ if $s \geq 2$ and $\delta \geq -\frac{3}{2}$.
- Setting $u = \phi - 1$, then Lichnerowicz equation (4.30) becomes

$$-\Delta_{\hat{h}} u = 2\pi \hat{z} (u+1)^{-3} + \frac{1}{8} \hat{K}_a^b \hat{K}_b^a (u+1)^{-7}. \quad (4.33)$$

- So applying Theorem 6.12 with $s' = s$ and $\delta' = \delta$ results that $(\phi - 1) = u \in H_{s,\delta}$ and $(\phi - 1) = u \geq 0$.

■

Combining our results of Section 4.1 with Theorem 4.3 we obtain the following corollary:

Corollary 4.4 (Construction of the data for the fluid) *Given the free data $(\bar{h}_{ab}, \bar{A}_{ab}, \hat{y}, \hat{v}^b)$ such that $(\bar{h}_{ab} - I) \in H_{s,\delta}$, $\bar{A}_{ab} \in H_{s-1,\delta+1}$, $(\hat{y}, \hat{v}^b) \in H_{s-1,\delta+2}$, $\frac{5}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$, $-\frac{3}{2} < \delta < -\frac{1}{2}$ and $(\hat{y}, \hat{v}^a) \in \Omega$, where Ω is given by (4.11). Then the data of the four velocity vector u^α and the Makino variable w are: $y = \phi^{-4(\gamma-1)}\hat{y}$, $v^b = \phi^{-2}\hat{v}^b$,*

$$(w, \bar{u}^a) := \Phi^{-1}(y, v^a) \quad \text{and} \quad \bar{u}^0 = 1 + h_{ab}\bar{u}^a\bar{u}^b$$

and the data for the energy and momentum densities are: $z = y^{\frac{2}{\gamma-1}}$, $j^a = zv^a$. These data satisfy the compatibility conditions (4.5) and (4.6). In addition, by Moser type estimate Theorem 8.11 and Proposition 8.8, $(w, \bar{u}^a) \in H_{s-1,\delta+2}$ and $\bar{u}^0 - 1 \in H_{s-1,\delta+2}$ and therefore if $\frac{7}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$, then these data have the needed regularity so they can serve as initial data for the hyperbolic system (2.9) for the perfect fluid.

4.3 Solutions to the elliptic systems

This section is devoted to the solutions the linear elliptic systems (4.24) and (4.28). The assumption on the given metric \bar{h}_{ab} is that $(\bar{h}_{ab} - I) \in H_{s,\delta}$. So according to Theorem 6.7 of Section 6, the operator $\Delta_{\bar{h}} : H_{s,\delta} \rightarrow H_{s-2,\delta+2}$ is semi Fredholm. In fact, it is an isomorphism, this can be shown in a similar manner to Step 1 of Section 6.3. We now consider the operator

$$L := -\Delta_{\bar{h}} + \frac{1}{8}R(\bar{h}) : H_{s,\delta} \rightarrow H_{s-2,\delta+2}, \quad (4.34)$$

which is also a semi Fredholm operator. If $R(\bar{h}) \geq 0$, then L is injective. However, a weaker condition is that L does not have non-positive eigenvalues, the variational formulation of this property known as the *Brill-Cantor condition* [8].

Let us first introduce few notations. For a Riemannian metric \bar{h}_{ab} , we set $(Du, Dv)_{\bar{h}} = \bar{h}^{ab}\partial_a u \partial_b v$, $|Du|_{\bar{h}}^2 = (Du, Du)_{\bar{h}}$ and $\mu_{\bar{h}}$ is the volume element with respect to the metric \bar{h}_{ab} .

Definition 4.5 (Brill-Cantor condition) *A metric \bar{h}_{ab} satisfies the Brill-Cantor condition if*

$$\inf_{u \neq 0} \frac{\int (|Du|_{\bar{h}}^2 + \frac{1}{8}R(\bar{h})u^2) d\mu_{\bar{h}}}{\int u^2 d\mu_{\bar{h}}} > 0, \quad (4.35)$$

where the infimum is taken over all $u \in C_0^1(\mathbb{R}^3)$.

This condition is invariant under conformal transformations, a fact that has been proved for example in [12]

Theorem 4.6 (Construction of a metric having zero scalar curvature) *Assume the given metric \bar{h}_{ab} satisfies $(\bar{h}_{ab} - \delta_{ab}) \in H_{s,\delta}$, $s \geq 2$, $\delta > -\frac{3}{2}$ and \bar{h}_{ab} satisfies the Brill-Cantor condition (4.35). Then there exists a scalar function α such that $\alpha - 1 \in H_{s,\delta}$, $\alpha(x) > 0$ and the metric $\hat{h}_{ab} = \alpha^4 \bar{h}_{ab}$ has a scalar curvature zero.*

Proof (of Theorem 4.6) The desired α is a solution to the elliptic equation (4.24). By setting $u = \alpha + 1$ this equation goes to

$$Lu = -\Delta_{\bar{h}}u + \frac{1}{8}R(\bar{h})u = -\frac{1}{8}R(\bar{h}). \quad (4.36)$$

We define for $\tau \in [0, 1]$, $L_\tau u = -\Delta_{\bar{h}}u + \frac{\tau}{8}R(\bar{h})u$. If $L_\tau u = 0$, then by Lemma 6.9, $u \in H_{s,-1}$ so

$$0 = (u, L_\tau u) = \int \left(|Du|_{\bar{h}}^2 + \frac{\tau}{8}R(\bar{h})u^2 \right) d\mu_{\bar{h}}. \quad (4.37)$$

Now, if $\int R(\bar{h})u^2 d\mu_{\bar{h}} \geq 0$, then obviously (4.37) implies that $u \equiv 0$. Otherwise $\int R(\bar{h})u^2 d\mu_{\bar{h}} < 0$, then there is sequence $\{u_n\} \subset C_0^\infty$ such that $u_n \rightarrow u$ in $H_{s,-1}$ - norm and

$$\int \left(|Du|_{\bar{h}}^2 + \frac{1}{8}R(\bar{h})u^2 \right) d\mu_{\bar{h}} = \lim_n \int \left(|Du_n|_{\bar{h}}^2 + \frac{1}{8}R(\bar{h})u_n^2 \right) d\mu_{\bar{h}} > 0 \quad (4.38)$$

by the Brill-Cantor condition (4.35). Substitution of (4.38) in (4.37) yields

$$0 = \int \left(|Du|_{\bar{h}}^2 + \frac{1}{8}R(\bar{h})u^2 \right) d\mu_{\bar{h}} + \frac{(\tau - 1)}{8} \int R(\bar{h})u^2 d\mu_{\bar{h}}, \quad (4.39)$$

this is certainly a contradiction. Thus L_τ is injective for each $\tau \in [0, 1]$, $L_0 = -\Delta_{\bar{h}}$ is isomorphism, hence $L_1 = -\Delta_{\bar{h}} + \frac{1}{8}R(\bar{h})$ is isomorphism by Theorem 6.8. Having proved the existence, we now show that $\alpha = u + 1$ is nonnegative. The set $\{x : \alpha(x) < 0\}$ has compact support since $\lim_{x \rightarrow \infty} u(x) = 0$ by the embedding Theorem 8.15. So letting $w = -\min(\alpha, 0)$, we have $w \in H_0^1(\mathbb{R}^3)$ and if the set $\{x : \alpha(x) < 0\}$ is not empty, then $w \not\equiv 0$ and then the Brill-Cantor condition gives

$$\int_{\{\alpha < 0\}} \left(|Dw|_{\bar{h}}^2 + \frac{1}{8}R(\bar{h})w^2 \right) d\mu_{\bar{h}} > 0. \quad (4.40)$$

On the other hand, according to Definition 6.10 of weak solutions,

$$0 = \int \left((D\alpha, Dw)_{\bar{h}} + \frac{1}{8}R(\bar{h})\alpha w \right) d\mu_{\bar{h}} = - \int_{\{\alpha < 0\}} \left(|Dw|_{\bar{h}}^2 + \frac{1}{8}R(\bar{h})w^2 \right) d\mu_{\bar{h}}. \quad (4.41)$$

So we conclude that $\alpha \geq 0$. Since $\alpha \geq 0$, we have by Harnack's inequality

$$\sup_{B_r} \alpha \leq C \inf_{B_r} \alpha$$

provided that B_r is sufficiently small ball. Hence, the set $\{\alpha(x) = 0\}$ is both open and closed, which is impossible. Thus $\alpha(x) > 0$. \blacksquare

Remark 4.7 *The conditions for applying Harnack's inequality to a second order elliptic operator*

$$Lu = \partial_a (A_{ab}(x) \partial_b u) + C(x)u$$

are boundedness of the coefficients (see e. g. [18]; Section 8) However, following carefully the proofs we found it can be applied also when the zero order coefficient belongs to $L^q_{\text{loc}}(\mathbb{R}^3)$ with $q > \frac{3}{2}$. In local coordinates equation (4.24) takes the form

$$L\alpha = \partial_a \left(\sqrt{|\bar{h}|} \bar{h}^{ab} \partial_b \alpha \right) + \sqrt{|\bar{h}|} R(\bar{h}) \alpha = 0.$$

For $s \geq 2$, $\sqrt{|\bar{h}|} \bar{h}^{ab}$ are bounded and non-degenerate, while $\sqrt{|\bar{h}|} R(\bar{h}) \in L^2_{\text{loc}}(\mathbb{R}^3)$.

We turn now the Lichnerowicz Laplacian system (4.28).

Theorem 4.8 (Solution of Lichnerowicz Laplacian) *Let \hat{h}_{ab} be a Riemannian metric in \mathbb{R}^3 so that $(\hat{h} - I) \in H_{s,\delta}$. Let vector $\hat{j}^b \in H_{s-2,\delta+2}$, $s \geq 2$ and $\delta > -\frac{3}{2}$. Then equation (4.28) has a unique solution $W \in H_{s,\delta}$.*

Proof (of theorem 4.8) In order to verify condition (H1) of Section 6.2 we compute the principle symbol of $L_{\Delta_{\hat{h}}}$ in (4.28). For each $\xi \in T_x^*M$, the principle symbol $(\Delta_{L_{\hat{h}}}(\xi))_a^b$ is a linear map from E_x to F_x , where E_x and F_x are fibers in T_xM . In local coordinates $\Delta_{\hat{h}} = \hat{h}^{ab} \partial_a \partial_b + \text{lower terms}$ and $D_a = \partial_a + \Gamma(\hat{h}^{ab}, \partial \hat{h}_{ab})$, hence

$$(\Delta_{L_{\hat{h}}}(\xi))_a^b = |\xi|_{\hat{h}}^2 \delta_a^b + \frac{1}{3} \xi^b \xi_a. \quad (4.42)$$

Therefore

$$((\Delta_{L_{\hat{h}}}(\xi)) \eta, \eta)_{\hat{h}} = \hat{h}^{bc} (\Delta_{L_{\hat{h}}}(\xi))_a^b \eta^a \eta^c = |\xi|_{\hat{h}}^2 |\eta|_{\hat{h}}^2 + \frac{1}{3} (\xi_a \eta^a)^2 \geq |\xi|_{\hat{h}}^2 |\eta|_{\hat{h}}^2. \quad (4.43)$$

Thus $(\Delta_{L_{\hat{h}}}(\xi))_a^b$ has positive eigenvalues and therefore $L_{\Delta_{\hat{h}}}$ is strongly elliptic. Furthermore, by Proposition 8.8 and Remark 8.12 we have that if $(\hat{h}_{ab} - I) \in H_{s,\delta}$, $s \geq 2$ and $\delta > -\frac{3}{2}$, then

$$\Delta_{L_{\hat{h}}} : H_{s,\delta} \rightarrow H_{s-2,\delta+2}.$$

Hence, we may apply Theorem 6.8 in order to obtain existence of the elliptic system (4.28). For the given metric \hat{h}_{ab} we define one parameter family of metrics $h_t = (1-t)I + t\hat{h}$, $0 \leq t \leq 1$, and the following associated operators with respect to these metrics: $(D_a)_t$ the covariant derivative, \mathcal{L}_t the Killing operator and $L_t = \Delta_{L_{h_t}} = (D)_t \cdot \mathcal{L}_t$ the Lichnerowicz Laplacian. We want to show that L_t is injective. We recall that $-2\mathcal{L}_t$ is the formal adjoint of D_t (see e. g. [3]), in addition, if $L_t(W) = 0$, then by Lemma 6.9 implies $W \in H_{s,-1}$. Thus we may use integration by parts and get

$$\begin{aligned} 0 &= (W, L_t W)_{h_t} = \int (h_t)_{ab} W^a L_t(W)^b d\mu_{h_t} = \int (h_t)_{ab} W^a (D_c)_t \cdot (\mathcal{L}_t W)^{cb} d\mu_{h_t} \\ &= -2 \int (h_t)_{ab} (h_t)_{dc} (\mathcal{L}_t W)^{ad} (\mathcal{L}_t W)^{cb} \mu_{h_t} = -2 \int |\mathcal{L}_t W|_{h_t}^2 \mu_{h_t} \end{aligned} \quad (4.44)$$

Now, if let $\tilde{h} = |h_t|^{-\frac{1}{3}}h_t$, then

$$\mathcal{L}_W \tilde{h} = |h_t|^{-\frac{1}{3}} \left(\mathcal{L}_W h_t - h_t \frac{2}{3} (D_a)_t W^a \right) = |h_t|^{-\frac{1}{3}} \mathcal{L}_t(W). \quad (4.45)$$

Thus $L_t(W) = \Delta_{L_{h_t}}(W) = 0$ implies $W \equiv 0$ if and only if there are no non-trivial Killing vector fields W in $H_{s,-1}$. This fact has been proved by G. Choquet and Y. Choquet-Bruhat [9] for $s > \frac{7}{2}$, D. Christodoulou and N. O’Murchadha for $s > 3 + \frac{3}{2}$ [14], and Bartnik for $s \geq 2$ [1] (See also Maxwell [30], where he obtained the minimum regularity $s > \frac{3}{2}$). Now $L_0 = \Delta_{L_I}$ is an operator with constant coefficients, so by Lemma 6.5 is an isomorphism. ■

5 Local Existence for Hyperbolic Equations

In this section we prove an existence theorem (locally in time) for quasi linear symmetric hyperbolic system in the $H_{s,\delta}$ spaces. The known existence results in the H^s space of Fisher and Marsden [16] and Kato [24] (see also [38], [37]), cannot be applied to the $H_{s,\delta}$ spaces. The main difficulty here is the establishment of energy estimates for linear hyperbolic systems. In order to achieve it we have defined a specific inner-product in $H_{s,\delta}$ (see Definition 5.3) and in addition the Kato-Ponce commutator estimate [25], [38], has an essential role in our approach. Once the energy estimates have been established in the $H_{s,\delta}$ space, we follow Majda’s [28] iteration procedure and show existence, uniqueness and continuity in that norm.

We consider the Cauchy problem for a quasi linear (uniform) symmetric hyperbolic system of the form

$$\begin{cases} A^0(u; t, x) \partial_t u + \sum_{a=1}^3 A^a(u; t, x) \partial_a u + B(u; t, x) u + F(u; t, x) = 0, \\ u(0, x) = u_0(x). \end{cases} \quad (5.1)$$

under the following assumptions:

- (H1) A^α are symmetric matrices for $\alpha = 0, 1, 2, 3$;
- (H2) $A^\alpha(u; t, x), B(u; t, x), F(u; t, x)$ are smooth in their arguments;
- (H3) $(A^0(0; t, \cdot) - I), A^a(0; t, \cdot), B(0; t, \cdot), F(0; t, \cdot) \in H_{s,\delta}$;
- (H4) $\partial_t A^0(u; t, \cdot) \in L^\infty$.

The main result of this section is the well posedness of the system (5.1) in $H_{s,\delta}$ spaces:

Theorem 5.1 (*Well posedness of first order hyperbolic symmetric systems in $H_{s,\delta}$*) Let $s > \frac{5}{2}$, $\delta \geq -\frac{3}{2}$ and assume hypotheses (H1)-(H4) holds. If the initial condition u_0 belongs to $H_{s,\delta}$ and satisfies

$$\frac{1}{\mu} \delta_{\alpha\beta} u_0^\alpha u_0^\beta \leq A_{\alpha\beta}^0 u_0^\alpha u_0^\beta \leq \mu \delta_{\alpha\beta} u_0^\alpha u_0^\beta, \quad \mu \in \mathbb{R}^+ \quad (5.2)$$

then there exists a positive T which depends on the $H_{s,\delta}$ -norm of the initial data and there exists a unique $u(t, x)$ a solution to (5.1) which in addition satisfies

$$u \in C([0, T], H_{s,\delta}) \cap C^1([0, T], H_{s-1,\delta+1}). \quad (5.3)$$

Remark 5.2 Condition (H3) is sometime too restrictive for applications. We may replace it by

$$(H3') \quad (A^0(U^0; t, \cdot) - I), A^a(U^0; t, \cdot), B(U^0; t, \cdot), F(U^0; t, \cdot) \in H_{s,\delta},$$

where U^0 is a constant vector. Setting $u = U^0 + v$, then v satisfies

$$\begin{cases} \tilde{A}^0(v; t, x) \partial_t v = \sum_{a=1}^3 \tilde{A}^a(u; t, x) \partial_a v + \tilde{B}(v; t, x) v + \tilde{F}(v; t, x) \\ v(0, x) = u_0(x) - U^0 \end{cases} \quad (5.4)$$

where $\tilde{A}^\alpha(v; t, x) = A^\alpha(U^0 + v; t, x)$, $\tilde{B}(v; t, x) = B(U^0 + v; t, x)$ and $\tilde{F}(v; t, x) = F(U^0 + v; t, x) + \tilde{B}(U^0 + v; t, x)$. The Moser type estimates are valid under assumptions (H3') (see Remark 8.12).

5.1 Strategy

We will proceed with the following strategy:

1. The establishment of energy estimates for linear systems in the fractional weighted spaces $H_{s,\delta}$.
2. We approximate the initial data by a C_0^∞ sequence and then construct an iteration process which consists of solutions to a linear system having a C_0^∞ initial data.
3. We show that the sequence which is constructed by the iteration process is bounded in $H_{s,\delta}$ -norm and weakly converges to a solution.
4. At the final stage we prove uniqueness and continuity in $H_{s,\delta}$ -norm.

5.2 Energy estimates in the fractional weighted spaces

The energy estimates are indispensable means for the proof of well posedness of hyperbolic systems. In order to achieve it we introduce an inner product which depends on a matrix A . We assume $A = A(t, x)$ is $m \times m$ symmetric matrix which satisfies

$$\frac{1}{\mu} U^T U \leq U^T A U \leq \mu U^T U \quad (5.5)$$

for some positive μ . Here B^T denotes the transpose matrix. We recall that $f_\epsilon(x) = f(\epsilon x)$, the sequence $\{\psi_j\}$ is a dyadic resolution of the unity in \mathbb{R}^3 which is defined the Appendix 7 and that $\Lambda^s u = \mathcal{F}^{-1}((1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} u)$, where \mathcal{F} denotes the Fourier transform. In this section the expression (5.6) below will serve as a norm of the space $H_{s,\delta}$:

$$\|u\|_{H_{s,\delta}}^2 := \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^2 u)_{(2j)}\|_{H^s}^2. \quad (5.6)$$

Corollary 7.6 implies that (5.6) is equivalent to the norm of Definition 3.1.

Definition 5.3 (Inner Product) For a symmetric matrix $A = A(t, x)$ which satisfies (5.5) we let

$$\begin{aligned} \langle u, v \rangle_{s,\delta,A} &:= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \langle \Lambda^s ((\psi_j^2 u)_{(2j)}), (A)_{2j} \Lambda^s ((\psi_j^2 v)_{(2j)}) \rangle_{L^2} \\ &= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \int [\Lambda^s ((\psi_j^2 u)_{(2j)})]^T (A)_{2j} [\Lambda^s ((\psi_j^2 v)_{(2j)})] dx \end{aligned} \quad (5.7)$$

and its associated norm $\|u\|_{H_{s,\delta,A}}^2 = \langle u, u \rangle_{s,\delta,A}$.

Obviously $\langle u, v \rangle_{s,\delta,A} = \langle v, u \rangle_{s,\delta,A}$ and from (5.5) we obtain the equivalence,

$$\frac{1}{\mu} \|u\|_{H_{s,\delta}}^2 \leq \|u\|_{H_{s,\delta,A}}^2 \leq \mu \|u\|_{H_{s,\delta}}^2. \quad (5.8)$$

We come now to the crucial estimate of this section.

Lemma 5.4 (An energy estimate) Let $s > \frac{5}{2}$, $\delta \geq -\frac{3}{2}$, $A^\alpha = A^\alpha(t, x)$ be $m \times m$ symmetric matrices such that $(A^0(t, \cdot) - I), A^\alpha(t, \cdot) \in H_{s,\delta}$ and A^0 satisfies (5.5). If $u(t) = u(t, \cdot)$ is a C_0^∞ solution of the linear hyperbolic system

$$A^0(t, x) \partial_t u = \sum_{a=1}^3 A^a(t, x) \partial_a u, \quad (5.9)$$

then

$$\frac{d}{dt} \|u(t)\|_{H_{s,\delta,A^0}}^2 \leq C \left(\mu \|u(t)\|_{H_{s,\delta,A^0}}^2 + 1 \right), \quad (5.10)$$

where $C = C(\|A^0 - I\|_{H_{s,\delta}}, \|A^a\|_{H_{s,\delta}}, \|\partial_t u\|_{H_{s-1,\delta}}, \|\partial_t A^0\|_{L^\infty})$.

An essential tool for deriving these estimates is the Kato & Ponce Commutator Estimate [25], [38].

Theorem 5.5 (Kato and Ponce) *Let $s > 0$, $f \in H^s \cap C^1$, $g \in H^{s-1} \cap L^\infty$, then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^2} \leq C \{\|\nabla f\|_{L^\infty} \|g\|_{H^{s-1}} + \|f\|_{H^s} \|g\|_{L^\infty}\}. \quad (5.11)$$

This estimate will be used term wise in the inner product (5.7).

Proof (of Lemma 5.4) Since u is C_0^∞ we may interchange the derivation with respect to t with the inner-product (5.7) and get

$$\begin{aligned} \frac{d}{dt} \langle u, u \rangle_{s,\delta,A^0} &= 2 \langle u, \partial_t u \rangle_{s,\delta,A^0} \\ &+ \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \int [\Lambda^s((\psi_j^2 u)_{(2j)})]^T (\partial_t A^0)_{2j} [\Lambda^s((\psi_j^2 u)_{(2j)})] dx \\ &\leq 2 \langle u, \partial_t u \rangle_{s,\delta,A^0} + \|\partial_t A^0\|_{L^\infty} \left(\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^2 u)_{(2j)}\|_{H^s}^2 \right) \\ &= 2 \langle u, \partial_t u \rangle_{s,\delta,A^0} + \|\partial_t A^0\|_{L^\infty} \|u\|_{H_{s,\delta}}^2 \end{aligned} \quad (5.12)$$

We turn now to the hard task of the proof, namely, the estimation of $\langle u, \partial_t u \rangle_{s,\delta,A^0}$. Put

$$E(j) = \left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), ((A^0)_{2j}) \Lambda^s \left((\psi_j^2 \partial_t u)_{2j} \right) \right\rangle_{L^2} \quad (5.13)$$

and let $\{\Psi_k\} = \frac{1}{\sum \psi_j(x)} \psi_k(x)$, where $\{\psi_j\}$ is defined in Appendix 7. It follows from the properties of this sequence that

$$\Psi_k \psi_j^2 \neq 0 \quad \text{only when } k = j - 3, \dots, j + 4. \quad (5.14)$$

Hence,

$$\begin{aligned} E(j) &= \left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), ((A^0)_{2j}) \Lambda^s \left(\left(\sum_{k=0}^{\infty} \Psi_k \right)_{2j} (\psi_j^2 \partial_t u)_{2j} \right) \right\rangle_{L^2} \\ &= \sum_{k=j-3}^{j+4} \left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), ((A^0)_{2j}) \Lambda^s \left((\Psi_k)_{2j} (\psi_j^2 \partial_t u)_{2j} \right) \right\rangle_{L^2} \\ &= \sum_{k=j-3}^{j+4} \left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), (A^0)_{2j} \left[\Lambda^s \left((\Psi_k)_{2j} (\psi_j^2 \partial_t u)_{2j} \right) - (\Psi_k)_{2j} \Lambda^s (\psi_j^2 \partial_t u)_{2j} \right] \right\rangle_{L^2} \\ &+ \sum_{k=j-3}^{j+4} \left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), (\Psi_k A^0)_{2j} \Lambda^s (\psi_j^2 \partial_t u)_{2j} \right\rangle_{L^2} \\ &= E_1(j, k) + E_2(j, k). \end{aligned}$$

This splitting will enable us to estimate $E_2(j, k)$ in terms of the $H_{s,\delta}$ norm of $A^0 - I$ while by Theorem 5.5,

$$\begin{aligned}
& |E_1(j, k)| \\
& \leq \left\| \Lambda^s \left((\psi_j^2 u)_{2j} \right) \right\|_{L^2} \|A^0\|_{L^\infty} \left\| \Lambda^s \left((\Psi_k)_{2j} (\psi_j^2 \partial_t u)_{2j} \right) - (\Psi_k)_{2j} \Lambda^s (\psi_j^2 \partial_t u)_{2j} \right\|_{L^2} \\
& \leq \left\| (\psi_j^2 u)_{2j} \right\|_{H^s} \|A^0\|_{L^\infty} \left\{ \|\nabla (\Psi_k)_{2j}\|_{L^\infty} \left\| (\psi_j^2 \partial_t u)_{2j} \right\|_{H^{s-1}} + \|(\Psi_k)_{2j}\|_{H^s} \left\| (\psi_j^2 \partial_t u)_{2j} \right\|_{L^\infty} \right\} \\
& \leq C \|A^0\|_{L^\infty} \left(\left\| (\psi_j^2 u)_{2j} \right\|_{H^s} \left\| (\psi_j^2 \partial_t u)_{2j} \right\|_{H^{s-1}} \right). \tag{5.15}
\end{aligned}$$

In the last step above we have used the below useful estimates. First, by (7.4) and (5.14),

$$\|\nabla (\Psi_k)_{2j}\|_{L^\infty} = 2^j \|\nabla \Psi_k\|_{L^\infty} \leq C 2^j 2^{-k} \leq 8C. \tag{5.16}$$

Secondly, it is well known that for any smooth function f ,

$$\|fu\|_{H^s} \leq C(\|f\|_{C^N})\|u\|_{H^s}, \tag{5.17}$$

where the integer N is not less than s . In addition, from (7.12) we see that

$$\|f_\epsilon\|_{H^s}^2 \lesssim \begin{cases} \epsilon^{-3} \|f\|_{H^s}^2, & \epsilon \leq 1 \\ \epsilon^{2s-3} \|f\|_{H^s}^2, & \epsilon \geq 1 \end{cases}. \tag{5.18}$$

Recalling that $\psi_k(x) = \psi_1(2^{-k}x)$ and $(\psi_k(x))_{2j} = (\psi_1(x))_{2j-k}$, applying the above and combining this with (5.14) and (5.17), we have

$$\begin{aligned}
\|(\Psi_k)_{2j}\|_{H^s} &= \left\| \left(\sum_j \psi_j \right)_{2j}^{-1} (\psi_k)_{2j} \right\|_{H^s} \leq C \|(\psi_k)_{2j}\|_{H^s} \\
&= C \left\| (\psi_1)_{2(j-k)} \right\|_{H^s} \leq C 2^{(s-\frac{3}{2})3} \|\psi_1\|_{H^s}. \tag{5.19}
\end{aligned}$$

Finally, by the Sobolev embedding

$$\|v\|_{L^\infty} \leq C \|v\|_{H^s}, \tag{5.20}$$

we obtain $\left\| (\psi_j^2 \partial_t u)_{2j} \right\|_{L^\infty} \leq C \left\| (\psi_j^2 \partial_t u)_{2j} \right\|_{H^{s-1}}$.

In order to use equation (5.9) we split $E_2(j, k)$ as follows:

$$\begin{aligned}
E_2(j, k) &= \left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), ((\Psi_k A^0)_{2j}) \Lambda^s \left((\psi_j^2 \partial_t u)_{2j} \right) \right\rangle_{L^2} \\
&= \left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), \left[(\Psi_k A^0)_{2j} \Lambda^s \left((\psi_j^2 \partial_t u)_{2j} \right) - \Lambda^s \left((\Psi_k A^0)_{2j} (\psi_j^2 \partial_t u)_{2j} \right) \right] \right\rangle_{L^2} \\
&+ \left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), \Lambda^s \left((\Psi_k A^0)_{2j} (\psi_j^2 \partial_t u)_{2j} \right) \right\rangle_{L^2} \\
&= E_3(j, k) + E_4(j, k).
\end{aligned}$$

In the estimation of the first term $E_3(j, k)$, the Kato-Ponce commutator estimate (5.11) is being used again:

$$\begin{aligned} & |E_3(j, k)| \\ & \leq C \left\| (\psi_j^2 u)_{2j} \right\|_{H^s} \left\{ \left\| \nabla (\Psi_k A^0)_{2j} \right\|_{L^\infty} \left\| (\psi_j^2 \partial_t u)_{2j} \right\|_{H^{s-1}} + \left\| (\Psi_k A^0)_{2j} \right\|_{H^s} \left\| (\psi_j^2 \partial_t u)_{2j} \right\|_{L^\infty} \right\}. \end{aligned}$$

From (5.16) and the embedding (5.20), we have

$$\begin{aligned} \left\| \nabla (\Psi_k A^0)_{2j} \right\|_{L^\infty} &= 2^j \left\| (\nabla (\Psi_k A^0 - I))_{2j} \right\|_{L^\infty} + 2^j \left\| \nabla (\Psi_k)_{2j} \right\|_{L^\infty} \\ &\leq C \left\{ 2^j \left\| (\nabla \Psi_k (A^0 - I))_{2j} \right\|_{H^{s-1}} + 1 \right\} \end{aligned}$$

and from (5.19)

$$\left\| (\Psi_k A^0)_{2j} \right\|_{H^s} \leq \left\| (\Psi_k (A^0 - I))_{2j} \right\|_{H^s} + \left\| \nabla (\Psi_k)_{2j} \right\|_{H^s} \leq \left\| (\Psi_k (A^0 - I))_{2j} \right\|_{H^s} + C.$$

Thus

$$\begin{aligned} & |E_3(j, k)| \\ & \leq C \left\| (\psi_j^2 u)_{2j} \right\|_{H^s} \left\| (\psi_j^2 \partial_t u)_{2j} \right\|_{H^{s-1}} \left\{ 2^j \left\| (\nabla \Psi_k (A^0 - I))_{2j} \right\|_{H^{s-1}} + 1 \right\} \\ & + C \left\| (\psi_j^2 u)_{2j} \right\|_{H^s} \left\| (\psi_j^2 \partial_t u)_{2j} \right\|_{L^\infty} \left\{ \left\| (\Psi_k (A^0 - I))_{2j} \right\|_{H^s} + 1 \right\} \\ & \leq C \left\| (\psi_j^2 u)_{2j} \right\|_{H^s} \left\| (\psi_j^2 \partial_t u)_{2j} \right\|_{H^{s-1}} \left\{ 2^j \left\| (\nabla \Psi_k (A^0 - I))_{2j} \right\|_{H^{s-1}} + 1 \right\} \quad (5.21) \\ & + C \left\| (\psi_j^2 u)_{2j} \right\|_{H^s} \left\| (\Psi_k (A^0 - I))_{2j} \right\|_{H^s} \left\| \partial_t u \right\|_{L^\infty} \\ & + C \left\| (\psi_j^2 u)_{2j} \right\|_{H^s} \left\| (\psi_j^2 \partial_t u)_{2j} \right\|_{H^{s-1}}. \end{aligned}$$

Now equation (5.9) is being utilized and

$$\begin{aligned} E_4(j, k) &= \left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), \Lambda^s \left((\Psi_k \psi_j^2)_{2j} (A^0 \partial_t u)_{2j} \right) \right\rangle_{L^2} \\ &= \left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), \Lambda^s \left((\Psi_k \psi_j^2)_{2j} \left(\sum_{a=1}^3 A^a \partial_a u \right)_{2j} \right) \right\rangle_{L^2} \\ &= \sum_{a=1}^3 \left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), \Lambda^s \left((\Psi_k A^a)_{2j} (\psi_j^2 \partial_a u)_{2j} \right) \right\rangle_{L^2} \quad (5.22) \\ &= \sum_{a=1}^3 \left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), \left[\Lambda^s \left((\Psi_k A^a)_{2j} (\psi_j^2 \partial_a u)_{2j} \right) - (\Psi_k A^a)_{2j} \Lambda^s \left((\psi_j^2 \partial_a u)_{2j} \right) \right] \right\rangle_{L^2} \\ &+ \sum_{a=1}^3 \left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), \left[(\Psi_k A^a)_{2j} \Lambda^s \left((\psi_j^2 \partial_a u)_{2j} \right) \right] \right\rangle_{L^2} \\ &= E_5(j, k, a) + E_6(j, k, a). \end{aligned}$$

Again, by Kato-Ponce commutator estimate (5.11),

$$\begin{aligned}
& |E_5(j, k, a)| \\
& \leq C \left\| (\psi_j^2 u)_{2j} \right\|_{H^s} \left\{ \left\| \nabla (\Psi_k A^a)_{2j} \right\|_{L^\infty} \left\| (\psi_j^2 \partial_a u)_{2j} \right\|_{H^{s-1}} + \left\| (\Psi_k A^a)_{2j} \right\|_{H^s} \left\| (\psi_j^2 \partial_a u)_{2j} \right\|_{L^\infty} \right\} \\
& \leq C \left\| (\psi_j^2 u)_{2j} \right\|_{H^s} \{ \left\| \nabla A^a \right\|_{L^\infty} + \|A^a\|_{L^\infty} \} 2^j \left\| (\psi_j^2 \partial_a u)_{2j} \right\|_{H^{s-1}} \\
& + C \left\| (\psi_j^2 u)_{2j} \right\|_{H^s} \|(\Psi_k A^a)_{2j}\|_{H^s} \|\partial_a u\|_{L^\infty}.
\end{aligned} \tag{5.23}$$

Using the commutation $\partial_a \Lambda^s = \Lambda^s \partial_a$ and the fact that $\Lambda^s (\psi_j^2 u)$ is rapidly decreasing, we see that

$$\begin{aligned}
0 &= \int \partial_a \left\{ \left[\Lambda^s \left((\psi_j^2 u)_{2j} \right) \right]^T (\Psi_k A^a)_{2j} \left[\Lambda^s \left((\psi_j^2 u)_{2j} \right) \right] \right\} dx \\
&= 2^j \int \left\{ \left[\Lambda^s \left((\psi_j^2 \partial_a u)_{2j} \right) \right]^T (\Psi_k A^a)_{2j} \left[\Lambda^s \left((\psi_j^2 u)_{2j} \right) \right] \right\} dx \\
&+ 2^j \int \left\{ \left[\Lambda^s \left((\psi_j^2 u)_{2j} \right) \right]^T (\Psi_k A^a)_{2j} \left[\Lambda^s \left((\psi_j^2 \partial_a u)_{2j} \right) \right] \right\} dx \\
&+ 2^j 2 \int \left\{ \left[\Lambda^s \left(((\partial_a \psi_j) \psi_j u)_{2j} \right) \right]^T (\Psi_k A^a)_{2j} \left[\Lambda^s \left((\psi_j^2 u)_{2j} \right) \right] \right\} dx \\
&+ 2^j 2 \int \left\{ \left[\Lambda^s \left((\psi_j^2 u)_{2j} \right) \right]^T (\Psi_k A^a)_{2j} \left[\Lambda^s \left(((\partial_a \psi_j) \psi_j u)_{2j} \right) \right] \right\} dx \\
&+ 2^j \int \left\{ \left[\Lambda^s \left((\psi_j^2 u)_{2j} \right) \right]^T (\partial_a (\Psi_k A^a))_{2j} \left[\Lambda^s \left((\psi_j^2 u)_{2j} \right) \right] \right\} dx.
\end{aligned}$$

Now $E_6(j, k, l)$ is equal to the second term, but since A^a is a symmetric matrix, the first and the second terms are equal and also the third and the forth. Hence by (5.17) and Cauchy Schwarz inequality,

$$\begin{aligned}
|2E_6(j, k, a)| &\leq 2 \|(\Psi_k A^a)_{2j}\|_{L^\infty} \|(\partial_a \psi_j \psi_j u)_{2j}\|_{H^s} \left\| (\psi_j^2 u)_{2j} \right\|_{H^s} \\
&+ \|(\partial_a (\Psi_k A^a))_{2j}\|_{L^\infty} \left\| (\psi_j^2 u)_{2j} \right\|_{H^s}^2 \\
&\leq C \|A^a\|_{L^\infty} \|(\psi_j u)_{2j}\|_{H^s} \left\| (\psi_j^2 u)_{2j} \right\|_{H^s} \\
&+ \{ \|\partial_a A^a\|_{L^\infty} + C \|A^a\|_{L^\infty} \} \left\| (\psi_j^2 u)_{2j} \right\|_{H^s}^2.
\end{aligned}$$

Taking the sum $\sum 2^{(\frac{3}{2}+\delta)2j} E(j)$ we are coming across three types of summations:

1. Given $v \in H_{s_1, \delta}$, $w \in H_{s_2, \delta}$ and γ_i equals 1 or 2, then

$$\begin{aligned}
& \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j^{\gamma_1} v)_{2j} \right\|_{H^{s_1}} \left\| (\psi_j^{\gamma_2} w)_{2j} \right\|_{H^{s_2}} \\
& \leq \frac{1}{2} \left(\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j^{\gamma_1} v)_{2j} \right\|_{H^{s_1}}^2 + 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j^{\gamma_2} w)_{2j} \right\|_{H^{s_2}}^2 \right) \\
& \leq C \left(\|v\|_{H_{s_1, \delta}}^2 + \|w\|_{H_{s_2, \delta}}^2 \right),
\end{aligned}$$

where in the last inequality the equivalence of the norms, (see Corollary 7.6), was involved.

2. Given $v \in H_{s, \delta}$ and $w \in H_{s, \delta}$, then from the scaling property (5.18) and (5.17) we have

$$\begin{aligned}
& \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j^2 v)_{2j} \right\|_{H^s} \left\| (\Psi_k w)_{2j} \right\|_{H^s} \\
& \leq \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j^2 v)_{2j} \right\|_{H^s}^2 + \frac{1}{2} \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \left\| (\Psi_k w)_{2j} \right\|_{H^s}^2 \\
& \leq \frac{7}{2} \|v\|_{H_{s, \delta}}^2 + C \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \left\| (\Psi_k w)_{2k} \right\|_{H^s}^2 \\
& \leq \frac{7}{2} \|v\|_{H_{s, \delta}}^2 + C \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2k} \left\| (\psi_k w)_{2k} \right\|_{H^s}^2 \\
& \leq C \left(7 \|v\|_{H_{s, \delta}}^2 + 7 \|w\|_{H_{s, \delta}}^2 \right).
\end{aligned}$$

3. Given $v \in H_{s_1, \delta}$, $w \in H_{s_2, \delta}$, $z \in H_{s_3, \delta}$ and γ_i equals 1 or 2, then by Hölder inequality,

Corollary 8.2 and the same arguments as in type 2, we get

$$\begin{aligned}
& \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^{\gamma_1} v)_{2j}\|_{H^{s_1}} \|(\psi_j^{\gamma_2} w)_{2j}\|_{H^{s_2}} 2^j \|(\nabla(\Psi_k z))_{2j}\|_{H^{s_3-1}} \\
& \leq \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)j} \|(\psi_j^{\gamma_1} v)_{2j}\|_{H^{s_1}} 2^{(\frac{3}{2}+\delta)j} \|(\psi_j^{\gamma_2} w)_{2j}\|_{H^{s_2}} 2^{(\frac{3}{2}+\delta+1)j} \|(\nabla(\Psi_k z))_{2j}\|_{H^{s_3-1}} \\
& \leq \left(\left(\sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} \left(2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^{\gamma_1} v)_{2j}\|_{H^{s_1}}^2 \right)^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
& \quad \times \left(\left(\sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} \left(2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^{\gamma_2} w)_{2j}\|_{H^{s_2}}^2 \right)^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
& \quad \times C \left(\sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta+1)2j} \|(\nabla(\psi_k z))_{2k}\|_{H^{s_3-1}}^2 \right)^{\frac{1}{2}} \\
& \leq \left(\sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^{\gamma_1} v)_{2j}\|_{H^{s_1}}^2 \right)^{\frac{1}{2}} \\
& \quad \times \left(\sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^{\gamma_2} w)_{2j}\|_{H^{s_2}}^2 \right)^{\frac{1}{2}} \\
& \quad \times C \left(\sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta+1)2k} \|(\nabla(\psi_k z))_{2k}\|_{H^{s_3-1}}^2 \right)^{\frac{1}{2}} \\
& \leq C \|v\|_{H_{s_1,\delta}} \|w\|_{H_{s_2,\delta}} \|\nabla z\|_{H_{s_3-1,\delta+1}} \\
& \leq C \|v\|_{H_{s_1,\delta}} \|w\|_{H_{s_2,\delta}} \|z\|_{H_{s_3,\delta}} \\
& \leq C \left(\|v\|_{H_{s_1,\delta}}^2 + \left(\|w\|_{H_{s_2,\delta}} \|z\|_{H_{s_3,\delta}} \right)^2 \right).
\end{aligned}$$

Applying these three types of inequalities we have,

$$\sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} |2E_6(j, k, a)| \leq C (\|A^a\|_{L^\infty} + \|\partial_a A^a\|_{L^\infty}) \|u\|_{H_{s,\delta}}^2, \quad (5.24)$$

$$\begin{aligned}
& \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} |E_5(j, k, a)| \\
& \leq C (\|\nabla A^a\|_{L^\infty} + \|A^a\|_{L^\infty}) \left\{ \|u\|_{H_{s,\delta}}^2 + \|\partial_a u\|_{H_{s-1,\delta+1}}^2 \right\} + C \left\{ \|u\|_{H_{s,\delta}}^2 + \|A^a\|_{H_{s,\delta}}^2 \|\partial_a u\|_{L^\infty}^2 \right\} \\
& \leq C (\|\nabla A^a\|_{L^\infty} + \|A^a\|_{L^\infty}) \left\{ \|u\|_{H_{s,\delta}}^2 + \|u\|_{H_{s,\delta}}^2 \right\} + C \left\{ \|u\|_{H_{s,\delta}}^2 + \|A^a\|_{H_{s,\delta}}^2 \|\partial_a u\|_{H_{s-1,\delta+1}}^2 \right\} \\
& \leq C \left\{ 2\|\nabla A^a\|_{L^\infty} + 2\|A^a\|_{L^\infty} + \|A^a\|_{H_{s,\delta}}^2 + 1 \right\} \|u\|_{H_{s,\delta}}^2, \tag{5.25}
\end{aligned}$$

here we have applied Corollary 8.2 and Theorem 8.15 to $\partial_a u$. Applying again Theorem 8.15 to $\|\partial_t u\|_{L^\infty}$ we have

$$\begin{aligned}
& \sum_{j=0}^{\infty} \sum_{k=j-3}^{j+4} 2^{(\frac{3}{2}+\delta)2j} |E_3(j, k)| \\
& \leq C \left\{ \|u\|_{H_{s,\delta}}^2 + \|\partial_t u\|_{H_{s-1,\delta}}^2 \|\nabla (A^0 - I)\|_{H_{s-1,\delta+1}}^2 \right\} + 2C \left\{ \|u\|_{H_{s,\delta}}^2 + \|\partial_t u\|_{H_{s-1,\delta}}^2 \right\} \\
& + C \left\{ \|u\|_{H_{s,\delta}}^2 + \|(A^0 - I)\|_{H_{s,\delta}}^2 \|\partial_t u\|_{L^\infty}^2 \right\} \tag{5.26} \\
& \leq 2C \left\{ \|u\|_{H_{s,\delta}}^2 + \|\partial_t u\|_{H_{s-1,\delta}}^2 \left(1 + \|A^0 - I\|_{H_{s,\delta}}^2 \right) \right\}
\end{aligned}$$

and finally

$$\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} |E_1(j)| \leq C \|A^0\|_{L^\infty} \left\{ \|u\|_{H_{s,\delta}}^2 + \|\partial_t u\|_{H_{s-1,\delta}}^2 \right\}. \tag{5.27}$$

Recalling that

$$\langle u, \partial_t u \rangle_{s,\delta,A^0} = \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} E(j) = \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), ((A^0)_{2j}) \Lambda^s \left((\psi_j^2 \partial_t u)_{2j} \right) \right\rangle_{L^2},$$

then inequalities (5.24), (5.25), (5.27) and (5.27) imply that

$$\langle u, \partial_t u \rangle_{s,\delta,A^0} \leq C (\|A^a\|_{L^\infty}, \|\nabla A^a\|_{L^\infty}, \|A^a\|_{H_{s,\delta}}, \|A^0 - I\|_{H_{s,\delta}}, \|\partial_t u\|_{H_{s-1,\delta}}) \left\{ \|u\|_{H_{s,\delta}}^2 + 1 \right\}.$$

Since $s > \frac{5}{2}$ and $\delta \geq -\frac{3}{2}$ we can use Theorem 8.15 (of the Appendix 8) and bound the norms $\|A^a\|_{L^\infty}$ and $\|\nabla A^a\|_{L^\infty}$ by the norms $\|A^0 - I\|_{H_{s,\delta}}$ and $\|A^a\|_{H_{s,\delta}}$. Thus, combining these bounds with above inequality and inequality (5.12), we have obtained

$$\frac{d}{dt} \langle u(t), u(t) \rangle_{s,\delta,A^0} \leq C \left(\|u(t)\|_{H_{s,\delta}}^2 + 1 \right), \tag{5.28}$$

where $C = C(\|A^a\|_{H_{s,\delta}}, \|A^0 - I\|_{H_{s,\delta}}, \|\partial_t u\|_{H_{s-1,\delta}}, \|\partial_t A^0\|_{L^\infty})$. Inserting the equivalence of norms $\|u\|_{H_{s,\delta}}^2 \leq \mu \|u\|_{H_{s,\delta,A^0}}^2$ in (5.28), we obtain (5.10) which completes the proof of Lemma 5.4. \blacksquare

We may extend the energy estimate (5.10) to a non-homogeneous symmetric hyperbolic systems.

Lemma 5.6 (An energy estimate) *Let $s > \frac{5}{2}$, $\delta \geq -\frac{3}{2}$, $A^\alpha = A^\alpha(t, x)$ be $m \times m$ symmetric matrices such that $(A^0(t, \cdot) - I), A^a(t, \cdot) \in H_{s, \delta}$ and A^0 satisfies (5.5). Let $B(t, \cdot), F(t, \cdot) \in H_{s, \delta}$. If $u(t, \cdot)$ is a C_0^∞ solution of the linear hyperbolic system*

$$A^0(t, x)\partial_t u = \sum_{a=1}^3 A^a(t, x)\partial_a u + B(t, x)u + F(t, x), \quad (5.29)$$

then

$$\frac{d}{dt} \|u(t)\|_{H_{s, \delta, A^0}}^2 \leq C \left(\mu \|u(t)\|_{H_{s, \delta, A^0}}^2 + 1 \right), \quad (5.30)$$

where the constant C depends on $\|A^a\|_{H_{s, \delta}}, \|A^0 - I\|_{H_{s, \delta}}, \|\partial_t u\|_{H_{s-1, \delta}}, \|\partial_t A^0\|_{L^\infty}, \|B\|_{H_{s, \delta}}$ and $\|F\|_{H_{s, \delta}}$.

Proof (of Lemma 5.6) This proof is precisely as the previous one expect the two terms

$$\left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), \Lambda^s \left((\Psi_k B)_{2j} (\psi_j^2 u)_{2j} \right) \right\rangle_{L^2} \quad (5.31)$$

and

$$\left\langle \Lambda^s \left((\psi_j^2 u)_{2j} \right), \Lambda^s \left((\Psi_k \psi_j^2 F)_{2j} \right) \right\rangle_{L^2} \quad (5.32)$$

which are added to (5.22). Using the algebra properties of H^s spaces, we see that (5.31) is less than

$$C \left\| (\psi_j^2 u)_{2j} \right\|_{H^s}^2 \|(\Psi_k B)_{2j}\|_{H^s} \leq C \left\| (\psi_j^2 u)_{2j} \right\|_{H^s}^2 \|B\|_{H_{s, \delta}};$$

and by Cauchy Schwarz inequality (5.32) is less than

$$C \left\| (\psi_j^2 u)_{2j} \right\|_{H^s} \left\| (\Psi_k \psi_j^2 F)_{2j} \right\|_{H^s} \leq C \frac{1}{2} \left\{ \left\| (\psi_j^2 u)_{2j} \right\|_{H^s}^2 + \left\| (\psi_j^2 F)_{2j} \right\|_{H^s}^2 \right\}.$$

Multiplying (5.31) and (5.32) by $2^{(\frac{3}{2} + \delta)2j}$ and taking the sum, it results with two quantities less than $\|u\|_{H_{s, \delta}}^2 \|B\|_{H_{s, \delta}}$ and $\left(\|u\|_{H_{s, \delta}}^2 + \|F\|_{H_{s, \delta}}^2 \right)$ respectively. \blacksquare

5.3 Construction of the iteration

We assume $u_0(x)$, the initial value of (5.1), is contained in G_1 , where the origin belongs to G_1 and G_1 is a compact subset of an open set G of \mathbb{R}^m . In addition we assume,

$$\frac{1}{\mu} U^T U \leq U^T A^0(U; t, x) U \leq \mu U^T U \quad \text{for all } U \in G_2, \quad (5.33)$$

where G_2 is a compact set of G such that $G_1 \Subset G_2$ and $\mu > 0$.

Remark 5.7 *Since the matrix A^0 is continuous, the initial condition (5.2) guarantees the existence of a domain G_2 .*

The initial data u_0 will be approximated by a sequence $\{u_0^k\}$ of smooth functions with compact support, which converges to u_0 in $H_{s,\delta}(\mathbb{R}^3)$. It follows from the embedding $\|v\|_{L^\infty} \leq C\|v\|_{H_{s,\delta}}$ (see Theorem 8.15, and the density Theorem 8.16, that there is a positive R , $u_0^0 \in C_0^\infty(\mathbb{R}^3)$ and $\{u_0^k\}_{k=1}^\infty \subset C_0^\infty(\mathbb{R}^3)$ such that

$$\|u_0^0\|_{H_{s+1,\delta}} \leq C\|u_0\|_{H_{s,\delta}}, \quad (5.34)$$

$$\|u_0^0 - u_0\|_{H_{s,\delta}} \leq \frac{R}{\mu 8}, \quad (5.35)$$

$$\|u - u_0^0\|_{H_{s,\delta}} \leq R \Rightarrow u \in G_2 \quad (5.36)$$

and

$$\|u_0^k - u_0\|_{H_{s,\delta}} \leq 2^{-k} \frac{R}{\mu 8}. \quad (5.37)$$

The iteration procedure is defined as follows: $u^0(t, x) = u_0^0(x)$ and $u^{k+1}(t, x)$ is a solution to the linear initial value problem

$$\begin{cases} A^0(u^k; t, x) \partial_t u^{k+1} = \sum_{a=1}^3 A^a(u^k; t, x) \partial_a u^{k+1} + B(u^k; t, x) u^{k+1} + F(u^k; t, x), \\ u^{k+1}(0, x) = u_0^{k+1}(x). \end{cases} \quad (5.38)$$

The existence of $\{u^k(t, x)\} \subset C_0^\infty(\mathbb{R}^3)$ follows from:

Theorem 5.8 (Existence of classical solutions of a linear symmetric hyperbolic system) *Let A^a , B and F be C^∞ functions and $v_0 \in C_0^\infty(\mathbb{R}^3)$ be an initial datum. Then the linear system*

$$\begin{cases} A^0(t, x) \partial_t v = \sum_{a=1}^3 A^a(t, x) \partial_a v + B(t, x) v + F(t, x) \\ v(0, x) = v_0(x) \end{cases} \quad (5.39)$$

has a unique solution $v(t, x)$ such that $v(t, x) \in C^\infty$ and it has compact support in \mathbb{R}^3 for each fixed t .

For the proof we refer to John [22]. It is evident from Theorem 5.8 and inequalities (5.33) and (5.36) that for each k : $u^k(t, x)$ is well defined, $u^k(t, x) \in C^\infty$, $u^k(t, x)$ has compact support in \mathbb{R}^3 and $u^k(t, x) \in G_2$ for some positive T . We put

$$T_k = \sup\{T : \sup_{0 < t < T} \|u^k(t) - u_0^0\|_{H_{s,\delta}} \leq R\}. \quad (5.40)$$

Our next issue is to show the existence of $T^* > 0$ such that $T_k \geq T^*$ for $k = 1, 2, 3, \dots$

5.4 Boundedness in the $H_{s,\delta}$ norm

We introduce the following notations: $u(t) := u(t, x)$ and

$$\|u\|_{s,\delta,T} := \sup\{\|u(t)\|_{H_{s,\delta}} : 0 \leq t \leq T\}. \quad (5.41)$$

The main result of this subsection is:

Lemma 5.9 (*Boundedness in the $H_{s,\delta}$ norm*) *There are positive constants T^* and L such that*

$$(A) \quad \|u^k - u_0^0\|_{s,\delta,T^*} \leq R$$

$$(B) \quad \|\partial_t u^k\|_{s-1,\delta+1,T^*} \leq L.$$

Proof (of Lemma 5.9) We first prove (B). Let

$$G^{k+1} = \sum_{a=1}^3 A^a(u^k; t, x) \partial_a u^{k+1} + B(u^k; t, x) u^{k+1} + F(u^k; t, x),$$

then by Proposition 8.8 and Moser type estimate, Theorem 8.11 with Remark 8.12,

$$\begin{aligned} & \|G^{k+1}\|_{H_{s-1,\delta+1}} \\ & \leq \sum_{a=1}^3 \|A^a(u^k)\|_{H_{s,\delta}} \|\partial_a u^k\|_{H_{s-1,\delta+1}} + \|B(u^k)\|_{H_{s,\delta}} \|u^k\|_{H_{s,\delta}} + \|F(u^k)\|_{H_{s,\delta}} \\ & \leq \sum_{a=1}^3 (C\|u^k\|_{H_{s,\delta}} + \|A^a(0)\|_{H_{s,\delta}}) \|u^k\|_{H_{s,\delta}} + (C\|u^k\|_{H_{s,\delta}} + \|B(0)\|_{H_{s,\delta}}) \|u^k\|_{H_{s,\delta}} \\ & \quad + C\|u^k\|_{H_{s,\delta}} + \|F(0)\|_{H_{s,\delta}}. \end{aligned} \quad (5.42)$$

The constant C here depends on $\|A^a\|_{C^{N+1}(G_2)}$, $\|B\|_{C^{N+1}(G_2)}$, $\|F\|_{C^{N+1}(G_2)}$ and $\|u^k\|_{L^\infty}$ (see (8.17)). Since

$$\|u^k(t)\|_{H_{s,\delta}} \leq \|u^k(t) - u_0^0\|_{H_{s,\delta}} + \|u_0^0\|_{H_{s,\delta}}, \quad (5.43)$$

the induction assumption (A) and inequality (5.34) imply that $\|u^k\|_{H_{s,\delta}} \leq R + C\|u_0\|_{H_{s,\delta}}$. Using the embedding $\|u^k\|_{L^\infty} \leq C\|u^k\|_{H_{s,\delta}}$, we see that $\|G^{k+1}\|_{H_{s-1,\delta+1}} \leq C_1(R)$, where the constant $C_1(R)$ depends upon R , condition (H3) and the initial data, but it is independent of k . From (5.38) we have

$$\partial_t u^{k+1} = (A^0(u^k; t, x))^{-1} G^{k+1} = \left((A^0(u^k; t, x))^{-1} - I \right) G^{k+1} + G^{k+1}.$$

Repeating same arguments as above, we conclude that

$$\left\| \left((A^0(u^k; t, x))^{-1} - I \right) G^{k+1} \right\|_{H_{s-1,\delta+1}} \leq C_2(R)$$

and the constant $C_2(R)$ does not depend on k . We take $L = C_1(R) + C_2(R)$. Here we have used Moser estimate with $F(u) = A^{-1}(u) - I$, and the formula $\frac{\partial A^{-1}(u)}{\partial u} = A^{-1}(u) \frac{\partial A(u)}{\partial u} A^{-1}(u)$. Thus the constant $C(R)$ depends on $\|A^0\|_{C^{N+2}(G_2)}$ and μ .

We turn now to show (A). Let $V^{k+1} = u^{k+1} - u_0^0$, then inserting it in the equation (5.38) we have obtained

$$\begin{aligned} A^0(u^k; t, x) \partial_t V^{k+1} &= A^0(u^k; t, x) u_t^{k+1} = \sum_{a=1}^3 A^a(u^k; t, x) \partial_a u^{k+1} + B(u^k; t, x) u^k + F(u^k; t, x) \\ &= \sum_{a=1}^3 A^a(u^k; t, x) \partial_a V^{k+1} + B(u^k; t, x) V^{k+1} + F(u^k; t, x) \\ &\quad + \sum_{a=1}^3 A^a(u^k; t, x) \partial_a u_0^0 + B(u^k; t, x) u_0^0 \end{aligned} \quad (5.44)$$

and $V^{k+1}(0, x) = u_0^{k+1}(0, x) - u_0^0(x)$. At this stage we would like employ the energy estimate Lemma 5.6. Due the fact that the coefficients of (5.44) depend on u^k , it is obligatory to control the constant of (5.30) in terms of $\|u^k\|_{H_{s,\delta}}$. Therefore we need to bound $\|A^0(u^k; t, x) - I\|_{H_{s,\delta}}$, $\|A^a(u^k; t, x)\|_{H_{s,\delta}}$, $\|B(u^k; t, x)\|_{H_{s,\delta}}$, $\|F(u^k; t, x)\|_{H_{s,\delta}}$ and $\|\frac{\partial}{\partial t} A^0(u^k; t, x)\|_{L^\infty}$ by $\|u^k\|_{H_{s,\delta}}$. The first four are similar, so take for example $A^a(u^k; t, x)$: We use assumption (H2), Moser type estimate, Theorem 8.11 with Remark 8.12 and get that

$$\|A^a(u^k; t, x)\|_{H_{s,\delta}} \leq C \left\{ \|A^a\|_{C^{N+1}(G_2)} (1 + \|u^k\|_{L^\infty}^N) \right\} \|u^k\|_{H_{s,\delta}} + \|A^a(0; t, \cdot)\|_{H_{s,\delta}}. \quad (5.45)$$

For the last one we have

$$\begin{aligned} \left\| \frac{\partial}{\partial t} A^0(u^k; t, x) \right\|_{L^\infty} &= \left\| \frac{\partial}{\partial u} A^0(u^k; t, x) \partial_t u^k(t, x) + \partial_t A^0(u^k; t, x) \right\|_{L^\infty} \\ &\leq \left\| \frac{\partial}{\partial u} A^0(u^k; t, x) \right\|_{L^\infty} \|\partial_t u^k(t, x)\|_{L^\infty} + \|\partial_t A^0(u^k; t, x)\|_{L^\infty} \\ &\leq C \left\| \frac{\partial}{\partial u} A^0(u^k; t, x) \right\|_{L^\infty} \|\partial_t u^k(t, x)\|_{H_{s-1,\delta+1}} + \|\partial_t A^0(u^k; t, x)\|_{L^\infty}. \end{aligned} \quad (5.46)$$

We conclude from inequalities (5.45) and (5.46), Theorem 8.15, the inductions hypothesis (A) and (B), (5.36) and (H4) that the constant of (5.30) depends on R , L , $\|u_0\|_{H_{s,\delta}}$ and the $H_{s,\delta}$ -norm of the coefficients, but it is independent of k . Hence, the energy estimate Lemma 5.6 implies that

$$\frac{d}{dt} \|V^{k+1}(t)\|_{H_{s,\delta,A^0}}^2 \leq C(R, L) \left(\mu \|V^{k+1}(t)\|_{H_{s,\delta,A^0}}^2 + 1 \right), \quad (5.47)$$

Applying Gronwall's inequality, (5.35), (5.37) and the equivalence (5.8) results in

$$\begin{aligned}
|||V^{k+1}|||_{s,\delta,T}^2 &\leq \mu e^{C(R,L)\mu T} \left(\mu \|V^{k+1}(0)\|_{H_{s,\delta}}^2 + C(R,L)T \right) \\
&= \mu e^{C(R,L)\mu T} \left(\mu \|u_0^{k+1} - u_0^0\|_{H_{s,\delta}}^2 + C(R,L)T \right) \\
&\leq \mu e^{C(R,L)\mu T} \left(\mu \left(\|u_0^{k+1} - u_0\|_{H_{s,\delta}}^2 + \|u_0^0 - u_0\|_{H_{s,\delta}}^2 \right) + C(R,L)T \right) \\
&\leq e^{C(R,L)\mu T} \left(2\mu^2 \left(\frac{R}{\mu 8} \right)^2 + \mu C(R,L)T \right). \tag{5.48}
\end{aligned}$$

Therefore $|||V^{k+1}|||_{s,\delta,T}^2 \leq R^2$, if $\mu C(R,L)T \leq \min\{\log 2, \frac{15}{32}R^2\}$. Thus taking $T^* = (\mu C(R,L))^{-1} \min\{\log 2, \frac{15}{32}R^2\}$ proves (A) and completes the proof of Lemma 5.9. \blacksquare

5.5 Contraction in the lower norm

We show here that $\{u^k\}$ has a contraction property in $\|\cdot\|_{0,\delta,T^{**}}$ for a positive T^{**} (see (5.41)). In order to achieved it we need an energy estimate in $H_{0,\delta} \sim L_\delta^2$. For that purpose we introduce the below inner-product in L_δ^2 : for two vectors u and v in L_δ^2 , we set

$$\langle u, v \rangle_{L_\delta^2, A^0} = \int (1 + |x|)^{2\delta} (u^T A^0 v) dx, \tag{5.49}$$

and its associated norm $\|u\|_{L_\delta^2, A^0}^2 = \langle u, u \rangle_{L_\delta^2, A^0}$. The ordinary norm is denoted by $\|u\|_{L_\delta^2}^2 = \langle u, u \rangle_{L_\delta^2, I}$. Since A^0 satisfies (5.33),

$$\frac{1}{\mu} \|u\|_{L_\delta^2}^2 \leq \langle u, u \rangle_{L_\delta^2, A^0} \leq \mu \|u\|_{L_\delta^2}^2, \tag{5.50}$$

and hence by Theorem we obtain 7.2, $\|u\|_{L_\delta^2, A^0}^2 \simeq \|u\|_{H_{0,\delta}}^2$.

Proposition 5.10 (Energy estimate in L_δ^2) *Suppose u satisfies the linear hyperbolic system (5.39), then*

$$\frac{d}{dt} \langle u(t), u(t) \rangle_{L_\delta^2, A^0} \leq \mu C \langle u(t), u(t) \rangle_{L_\delta^2, A^0} + \|F\|_{L_\delta^2}^2, \tag{5.51}$$

where $C = C(\|\partial_t A^0\|_{L^\infty}, \|A^a\|_{L^\infty}, \|B\|_{L^\infty}, \|\partial_a A^a\|_{L^\infty})$.

Proof (of Proposition 5.10) Taking the derivative of (5.49) with respect to t , we get

$$\begin{aligned}
\frac{d}{dt} \langle u, u \rangle_{L_\delta^2, A^0} &= 2 \langle u, \partial_t u \rangle_{L_\delta^2, A^0} + \int (1 + |x|)^{2\delta} (u^T \partial_t A^0 u) dx \\
&= 2 \sum_{a=1}^3 \int (1 + |x|)^{2\delta} (u^T A^a \partial_a u) dx + 2 \int (1 + |x|)^{2\delta} (u^T B u) dx \\
&+ 2 \int (1 + |x|)^{2\delta} (u^T F) dx + \int (1 + |x|)^{2\delta} (u^T \partial_t A^0 u) dx \\
&= 2 \sum_{a=1}^3 L_{1,a} + 2L_2 + 2L_3 + L_4.
\end{aligned}$$

Clearly,

$$|L_2| \leq \|B\|_{L^\infty} \int (1 + |x|)^{2\delta} |u|^2 dx \leq \|B\|_{L^\infty} \|u\|_{L_\delta^2}^2$$

and in a similar way we obtain the estimates of L_4 while by Cauchy-Schwarz inequality,

$$|L_3| \leq \|u\|_{L_\delta^2} \|F\|_{L_\delta^2} \leq \frac{1}{2} \left(\|u\|_{L_\delta^2}^2 + \|F\|_{L_\delta^2}^2 \right).$$

Now,

$$\begin{aligned}
0 &= \int \partial_a \left((1 + |x|)^{2\delta} (u^T A^a u) \right) dx \\
&= 2\delta \int (1 + |x|)^{2\delta-1} \frac{x_a}{|x|} (u^T A^a u) dx + \int (1 + |x|)^{2\delta} ((\partial_a u)^T A^a u) dx \\
&+ \int (1 + |x|)^{2\delta} (u^T \partial_a A^a u) dx + \int (1 + |x|)^{2\delta} (u^T A^a \partial_a u) dx,
\end{aligned}$$

and since A^0 is symmetric, the second and the fourth terms are equal to $L_{1,a}$. Hence,

$$\begin{aligned}
2|L_{1,a}| &\leq 2\delta \int (1 + |x|)^{2\delta} \frac{|A^0|}{1 + |x|} (|u|^2) dx + \int (1 + |x|)^{2\delta} |\partial_a A^a| |u|^2 dx \\
&\leq (\|A^a\|_{L^\infty} + \|\partial_a A\|_{L^\infty}) \|u\|_{L_\delta^2}^2.
\end{aligned}$$

■

In order to proof the contraction we shall also need the following proposition.

Proposition 5.11 (Difference estimate in L_δ^2) Let $G : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a C^1 mapping. Then

$$\|G(u) - G(v)\|_{L_\delta^2}^2 \leq \|\nabla G\|_{L^\infty}^2 \|u - v\|_{L_\delta^2}^2. \quad (5.52)$$

Proof (of Proposition 5.11)

$$\begin{aligned} \|G(u) - G(v)\|_{L_\delta^2}^2 &= \int (1 + |x|)^{2\delta} (G(u) - G(v))^2 dx \\ &= \int (1 + |x|)^{2\delta} \left(\int_0^1 \nabla G(su + (1-s)v) (u-v) ds \right)^2 dx \leq \|\nabla G\|_{L^\infty}^2 \|u - v\|_{L_\delta^2}^2. \end{aligned}$$

■

Lemma 5.12 (Contraction in a lower norm) *There is a positive T^{**} , $0 < \Lambda < 1$ and a positive sequence $\{\beta_k\}$ with $\sum \beta_k < \infty$ such that*

$$\|u^{k+1} - u^k\|_{0,\delta,T^{**}} \leq \Lambda \|u^k - u^{k-1}\|_{0,\delta,T^{**}} + \beta_k. \quad (5.53)$$

Here $\|u\|_{0,\delta,T^{**}} = \sup\{\|u(t)\|_{H_{0,\delta}} : 0 \leq t \leq T^{**}\}$.

Proof (of Lemma 5.12) Since u^k satisfies equation (5.38), the difference $[u^{k+1} - u^k]$ will satisfy

$$\begin{aligned} A^0(u^k; t, x) \partial_t [u^{k+1} - u^k] &= \sum_{a=1}^3 A^a(u^k; t, x) \partial_a [u^{k+1} - u^k] \\ &+ B(u^k; t, x) [u^{k+1} - u^k] + F^k, \end{aligned} \quad (5.54)$$

where

$$\begin{aligned} F^k &= -[A^0(u^k; t, x) - A^0(u^{k-1}; t, x)] \partial_t u^k + \sum_{a=1}^3 [A^a(u^k; t, x) - A^a(u^{k-1}; t, x)] \partial_a u^k \\ &+ [B(u^k; t, x) - B(u^{k-1}; t, x)] u^k + [F(u^k; t, x) - F(u^{k-1}; t, x)]. \end{aligned}$$

Applying Proposition 5.10 to equation (5.54) above we have

$$\frac{d}{dt} \langle [u^{k+1} - u^k], [u^{k+1} - u^k] \rangle_{L_\delta^2, A^0} \leq \mu C \langle [u^{k+1} - u^k], [u^{k+1} - u^k] \rangle_{L_\delta^2, A^0} + \|F^k\|_{L_\delta^2}^2. \quad (5.55)$$

Thus Gronwall's inequality yields,

$$\| [u^{k+1}(t) - u^k(t)] \|_{L_\delta^2, A^0}^2 \leq e^{\mu C t} \left[\| [u^{k+1}(0) - u^k(0)] \|_{L_\delta^2, A^0}^2 + \int_0^t \|F^k(s)\|_{L_\delta^2}^2 ds \right]. \quad (5.56)$$

The constant C in inequalities (5.55) and (5.56) depends on $\|A^a(u^k; t, x)\|_{L^\infty}$, $\|B(u^k; t, x)\|_{L^\infty}$, $\|\partial_t(A^0(u^k; t, x))\|_{L^\infty}$ and $\|\partial_a(A^a(u^k; t, x))\|_{L^\infty}$. The first two of them are bounded by a constant independent of k , since it follows from (A) of Lemma 5.9 that

$u^k \in G_2$. The estimation of $\|\partial_t(A^0(u^k; t, x))\|_{L^\infty}$ is done in (5.46) and for the last one, since $s - 1 > \frac{3}{2}$, we can use Theorem 8.15 and Corollary 8.2 and get

$$\begin{aligned} \|\partial_a(A^a(u^k; t, x))\|_{L^\infty} &\leq \left\| \frac{\partial}{\partial u} A^a(u^k; t, x) \partial_a u^k \right\|_{L^\infty} + \|\partial_a A^a(u^k; t, x)\|_{L^\infty} \\ &\leq C \left\| \frac{\partial}{\partial u} A^a(u^k; t, x) \right\|_{L^\infty} \|\partial_a u^k\|_{H_{s-1, \delta+1}} + \|\partial_a A^a(u^k; t, x)\|_{L^\infty} \\ &\leq C \left\| \frac{\partial}{\partial u} A^a(u^k; t, x) \right\|_{L^\infty} \|u^k\|_{H_{s, \delta}} + \|\partial_a A^a(u^k; t, x)\|_{L^\infty} \end{aligned}$$

Lemma 5.9 (A) implies that $\|u^k\|_{H_{s, \delta}}$ is bounded and $u^k \in G_2$, therefore the above inequality shows that $\|\partial_a(A^a(u^k; t, x))\|_{L^\infty}$ is bounded by a constant independent of k . From Proposition 5.11 we obtain

$$\begin{aligned} \|F^k\|_{L_\delta^2}^2 &\leq 2 \left\{ \|\nabla A^0\|_{L^\infty(G_2)}^2 \|\partial_t u^k\|_{L^\infty}^2 + \sum_{a=1}^3 \|\nabla A^a\|_{L^\infty(G_2)}^2 \|\partial_a u^k\|_{L^\infty}^2 \right. \\ &\quad \left. + \|\nabla B\|_{L^\infty(G_2)}^2 \|u^k\|_{L^\infty}^2 + \|\nabla F\|_{L^\infty(G_2)}^2 \right\} \|u^k - u^{k-1}\|_{L_\delta^2}^2, \end{aligned} \quad (5.57)$$

here ∇ is the gradient with respect to u . Since by Theorem 8.15 and Corollary 8.2, $\|\partial_t u^k\|_{L^\infty} \leq C \|\partial_t u^k\|_{H_{s-1, \delta+1}}$, $\|\partial_a u^k\|_{L^\infty} \leq C \|u^k\|_{H_{s-1, \delta+1}} \leq C \|u^k\|_{H_{s, \delta}}$ and $\|u^k\|_{L^\infty} \leq C \|u^k\|_{H_{s, \delta}}$, it follows from (5.57) and Lemma 5.9 that

$$\|F^k(s)\|_{L_\delta^2}^2 \leq C_1 \|u^k(s) - u^{k-1}(s)\|_{L_\delta^2}^2, \quad (5.58)$$

where the constant C_1 depends upon R and L of Lemma 5.9, but it is independent of k . By the equivalence $\|u\|_{L_\delta^2, A^0}^2 \simeq \|u\|_{H_{0, \delta}}^2$, (5.58) and (5.56) above, we conclude that

$$\begin{aligned} &\|u^{k+1}(t) - u^k(t)\|_{H_{0, \delta}}^2 \\ &\leq C_2 e^{\mu C t} \left[\|u^{k+1}(0) - u^k(0)\|_{H_{0, \delta}}^2 + C_1 \int_0^t \|u^k(s) - u^{k-1}(s)\|_{H_{0, \delta}}^2 ds \right] \\ &\leq C_2 e^{\mu C t} \left[\|u^{k+1}(0) - u^k(0)\|_{H_{0, \delta}}^2 + C_1 t \sup_{0 \leq s \leq t} \|u^k(s) - u^{k-1}(s)\|_{H_{0, \delta}}^2 \right], \end{aligned}$$

where C_1 , C_2 and C do not depend on k . Hence

$$\begin{aligned} &\|u^{k+1} - u^k\|_{0, \delta, T^{**}} \\ &\leq \sqrt{\frac{C_2 e^{\mu C T^{**}}}{2}} \|u^{k+1}(0) - u^k(0)\|_{H_{0, \delta}} + \sqrt{\frac{C_1 e^{\mu C T^{**}}}{2}} \|u^k - u^{k-1}\|_{0, \delta, T^{**}}. \end{aligned}$$

Thus, taking T^{**} sufficiently small so that $\Lambda := \sqrt{\frac{C_1 e^{\mu C T^{**}}}{2}} < 1$ and putting $\beta_k = \sqrt{\frac{C_2 e^{\mu C T^{**}}}{2}} \|u^{k+1}(0) - u^k(0)\|_{H_{0, \delta}}$ completes the proof of the Lemma. \blacksquare

Lemma 5.12 implies that $\{u^k\}$ is a Cauchy sequence in $C([0, T^{**}], H_{0,\delta})$. Combing this with the intermediate estimates $\|u\|_{H_{s',\delta}} \leq \|u\|_{H_{s,\delta}}^{\frac{s'}{s}} \|u\|_{H_{0,\delta}}^{1-\frac{s'}{s}}$ (see Proposition 8.7) and Lemma 5.9 (A), we conclude that $\{u^k\}$ is a Cauchy sequence in $C([0, T^{**}], H_{s',\delta})$ for any $s' < s$. Therefore there is a unique $u \in C([0, T^{**}], H_{s',\delta})$ such that

$$\|u^k - u\|_{s',\delta,T^{**}} \rightarrow 0 \quad \text{for any} \quad s' < s. \quad (5.59)$$

Taking $\frac{5}{2} < s' < s$ and utilizing the embedding Theorem 8.15, we have

$$u^k \rightarrow u \quad \text{in} \quad C([0, T^{**}], C_{\beta}^1(\mathbb{R}^3)) \quad \text{for any} \quad \beta \leq \delta + \frac{3}{2},$$

where $C_{\beta}^1(\mathbb{R}^3)$ is the class for which the norm

$$\sup_{\mathbb{R}^3} \left((1 + |x|)^{\beta} |u(x)| + \sum_{a=1}^3 (1 + |x|)^{\beta+1} |\partial_a u(x)| \right)$$

is finite. From (5.38)

$$\partial_t u^{k+1} = (A^0(u^k; t, x))^{-1} \left[\sum_{a=1}^3 A^a(u^k; t, x) \partial_a u^{k+1} + B(u^k; t, x) u^{k+1} + F(u^k; t, x) \right],$$

therefore by Corollary 8.13 and Proposition 8.7, we obtain $\partial_t u^k \rightarrow \partial_t u$ in $H_{s-1,\delta+1}$. Hence

$$\partial_t u^k \rightarrow \partial_t u \quad \text{in} \quad C([0, T^{**}], C_{\beta+1}(\mathbb{R}^3)) \quad \text{for any} \quad \beta \leq \delta + \frac{3}{2}.$$

Thus $u \in C^1(\mathbb{R}^3 \times [0, T^{**}])$ is a classical solution of the nonlinear system (5.1). Moreover, it follows from Lemma 5.9 (B) that $u \in \text{Lip}([0, T^{**}], H_{s-1,\delta+1})$. Our next task is to show that u^k converges weakly to u in $H_{s,\delta}$.

5.6 Weak Convergence

We first define the standard inner-product on $H_{s,\delta}$. For two vector valued functions $v, \phi \in H^s$, the expression

$$\langle v, \phi \rangle_s = \int (\Lambda^s v)^T \Lambda^s(\phi) dx = \int (1 + |\xi|^2)^s \hat{v}^T \overline{\hat{\phi}} d\xi.$$

is an inner-product on H^s . Utilizing Definition 3.1 and Corollary 7.6 we see that

$$\langle v, \phi \rangle_{s,\delta} = \sum_j 2^{(\frac{3}{2}+\delta)2j} \langle (\psi_j^2 v)_{(2j)}, (\psi_j^2 \phi)_{(2j)} \rangle_s$$

is inner-product on $H_{s,\delta}$. This definition coincides with Definition 5.3 in the case where A is the identity matrix.

Lemma 5.13 (Weak Convergence) For any $\phi \in H_{s,\delta}$, we have

$$\lim_k \langle u^k(t), \phi \rangle_{s,\delta} = \langle u(t), \phi \rangle_{s,\delta} \quad (5.60)$$

uniformly for $0 \leq t \leq T^{**}$. Consequently

$$\|u(t)\|_{H_{s,\delta}} \leq \liminf_k \|u^k(t)\|_{H_{s,\delta}} \quad (5.61)$$

and hence the solution u of the initial value problem (5.1) belongs to $L^\infty([0, T^{**}], H_{s,\delta})$.

In order to show Lemma 5.13 we need the below property.

Proposition 5.14 Let $s < \frac{s' + s''}{2}$, $v \in H_{s',\delta}$, $\phi \in H_{s'',\delta}$. Then we have

$$|\langle v, \phi \rangle_{s,\delta}| \leq \|v\|_{H_{s',\delta}} \|\phi\|_{H_{s'',\delta}}. \quad (5.62)$$

Proof (of Proposition 5.14) Elementary arguments show that

$$|\langle v, \phi \rangle_s| \leq \|v\|_{H^{s'}} \|\phi\|_{H^{s''}}.$$

Applying it term-wise and using the Cauchy-Schwarz inequality we have

$$\begin{aligned} |\langle v, \phi \rangle_{s,\delta}| &\leq \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left| \left\langle (\psi_j^2 v)_{2j}, (\psi_j^2 \phi)_{2j} \right\rangle_s \right| \\ &\leq \sum_{j=0}^{\infty} \left(2^{(\frac{3}{2}+\delta)j} \left\| (\psi_j^2 v)_{2j} \right\|_{H^{s'}} \right) \left(2^{(\frac{3}{2}+\delta)j} \left\| (\psi_j^2 \phi)_{2j} \right\|_{H^{s''}} \right) \\ &\leq \left(\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j^2 v)_{2j} \right\|_{H^{s'}}^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j^2 \phi)_{2j} \right\|_{H^{s''}}^2 \right)^{\frac{1}{2}} \\ &= \|v\|_{H_{s',\delta}} \|\phi\|_{H_{s'',\delta}} \end{aligned}$$

■

Proof (of Lemma 5.13) Take s' and s'' such that $s' < s < s''$ and $s < \frac{s' + s''}{2}$. For a given $\phi \in H_{s,\delta}$ and positive ϵ , we may find by Theorem 8.16 (b), $\tilde{\phi} \in H_{s'',\delta}$ such that

$$\|\phi - \tilde{\phi}\|_{H_{s,\delta}} \leq \frac{\epsilon}{2R} \quad \text{and} \quad \|\tilde{\phi}\|_{H_{s'',\delta}} \leq C(\epsilon) \|\phi\|_{H_{s,\delta}}, \quad (5.63)$$

where R is the positive number appearing in (5.36). Now,

$$\begin{aligned} \langle u^k(t) - u(t), \phi \rangle_{s,\delta} &= \left\langle u^k(t) - u(t), \tilde{\phi} \right\rangle_{s,\delta} + \left\langle u^k(t) - u(t), (\phi - \tilde{\phi}) \right\rangle_{s,\delta} \\ &= I_k + II_k. \end{aligned}$$

Therefore Proposition 5.14, (5.63) and (5.59) imply that

$$|I_k| \leq \|u^k(t) - u(t)\|_{H_{s',\delta}} \|\tilde{\phi}\|_{H_{s'',\delta}} \leq \|u^k(t) - u(t)\|_{H_{s',\delta}} C(\epsilon) \|\phi\|_{H_{s,\delta}} \rightarrow 0.$$

While in the second estimate we use Lemma 5.9 (A) and get

$$\begin{aligned} |II_k| &\leq \|u^k(t) - u(t)\|_{H_{s,\delta}} \|\phi - \tilde{\phi}\|_{H_{s,\delta}} \\ &\leq (\|u^k(t) - u_0^0\|_{H_{s,\delta}} + \|u(t) - u_0^0\|_{H_{s,\delta}}) \|\phi - \tilde{\phi}\|_{H_{s,\delta}} \leq \frac{2R\epsilon}{2R} = \epsilon. \end{aligned}$$

Thus,

$$\limsup_k \left| \langle u^k(t) - u(t), \phi \rangle_{s,\delta} \right| \leq \epsilon$$

which completes the proof of the limit (5.60). \blacksquare

For each k , $\langle u^k(t), \phi \rangle_{s,\delta}$ is continuous for $t \in [0, T^{**}]$ and by Lemma 5.13 it converges uniformly to $\langle u(t), \phi \rangle_{s,\delta}$, hence $\langle u(t), \phi \rangle_{s,\delta}$ is a continuous function of t for any $\phi \in H_{s,\delta}$ and we have obtained the following:

Theorem 5.15 (Existence) *Under conditions (H1)-(H4) and (5.33) there is $u \in C^1(\mathbb{R}^3 \times [0, T^{**}])$ a classical solution to the hyperbolic system (5.1) such that $u(t, x) \in \overline{G_2}$ and*

$$u \in L^\infty([0, T^{**}], H_{s,\delta}) \cap C_w([0, T^{**}], H_{s,\delta}) \cap \text{Lip}([0, T^{**}], H_{s-1,\delta+1}), \quad (5.64)$$

where C_w means continuous in the weak topology of $H_{s,\delta}$.

5.7 Well-posedness

In this subsection we will prove continuity in $H_{s,\delta}$ -norm and uniqueness.

Theorem 5.16 (Uniqueness) *Assume conditions (H1)-(H4) and (5.33) hold. If $u_1(t, x)$ and $u_2(t, x)$ are classical solutions to the hyperbolic system (5.1) such that $u_1, u_2 \in \overline{G_2}$, then $u_1 \equiv u_2$.*

Proof (of Theorem 5.16) Let u_1 and u_2 be a solutions to the hyperbolic system hyperbolic system (5.1) with the same initial data and let $V(t, x) = u_1(t, x) - u_2(t, x)$. Then V satisfies the equation

$$A^0(u_1; t, x) \partial_t V = \sum_{a=1}^3 A^a(u_1; t, x) \partial_a V + B(u_1; t, x) V + G \quad (5.65)$$

with the initial condition $V(0, x) = 0$ and where

$$\begin{aligned} G &= [A^0(u_1; t, x) - A^0(u_2)] \partial_t u_1 + \sum_{a=1}^3 [A^a(u_1; t, x) - A^a(u_2; t, x)] \partial_a u_1 \\ &\quad + [B(u_1; t, x) - B(u_2; t, x)] u_1 + [F(u_1; t, x) - F(u_2; t, x)]. \end{aligned}$$

Applying Proposition 5.10 to (5.65), we have

$$\frac{d}{dt} \langle V, V \rangle_{L^2_\delta, A^0(u_1)} \leq \mu C \langle V, V \rangle_{L^2_\delta, A^0(u_1)} + \|G\|_{L^2_\delta}^2.$$

Let $T \leq T^*$, then Gronwall's inequality and the equivalence (5.50) imply

$$\|V\|_{0,\delta,T}^2 \leq C_1 e^{C\mu T} \int_0^T \|G(t)\|_{L^2_\delta}^2 dt.$$

Similar estimation as done in (5.57) yield that $\|G(t)\|_{L^2_\delta}^2 \leq C_2 \|V(t)\|_{L^2_\delta}^2$. Hence,

$$\|V\|_{0,\delta,T}^2 \leq C_3 e^{C\mu T} T \|V\|_{0,\delta,T}^2. \quad (5.66)$$

Thus, if T is sufficiently small, then (5.66) leads to a contradiction unless $V \equiv 0$. \blacksquare

Theorem 5.17 (Continuation in norm) *Under conditions (H1)-(H4) and (5.33), any solutions u to the hyperbolic system (5.1) which satisfies $u(t, x) \in \overline{G_2}$ and the regularity condition (5.64), satisfies in addition*

$$u \in C([0, T^{**}], H_{s,\delta}) \cap C^1([0, T^{**}], H_{s-1,\delta+1}). \quad (5.67)$$

Proof (of Theorem 5.17) We first treat the continuity $C([0, T^{**}], H_{s,\delta})$. Since u is a solution of initial value problem (5.1) which is reversible in time, is sufficient to show that

$$\lim_{t \downarrow 0} \|u(t) - u(0)\|_{H_{s,\delta}} = \lim_{t \downarrow 0} \|u(t) - u_0\|_{H_{s,\delta}} = 0. \quad (5.68)$$

We shall use the following known argument: suppose $\{w_n\}$ is a sequence in Hilbert space which converge weakly to w_0 and $\limsup_n \|w_n\| \leq \|w_0\|$, then $\lim_n \|w_n - w_0\| = 0$. We are going to use the equivalence norm $\|\cdot\|_{H_{s,\delta,A^0(u(0))}}$, so we need to show

$$\limsup_{t \downarrow 0} \|u(t)\|_{H_{s,\delta,A^0(u(0))}} \leq \|u_0\|_{H_{s,\delta,A^0(u(0))}}. \quad (5.69)$$

Let $\{u^k(t)\}$ be the sequence which is defined by the iteration process (5.38). It follows from the uniqueness Theorem 5.16 and (5.61) that

$$\|u(t)\|_{H_{s,\delta,A^0(u(t))}} \leq \liminf_k \|u^k(t)\|_{H_{s,\delta,A^0(u(t))}}, \quad (5.70)$$

where the limit above is uniformly in t . Applying the energy estimate (5.30), we have

$$\frac{d}{dt} \|u^{k+1}(t)\|_{H_{s,\delta,A^0(u^k(t))}}^2 \leq C \left(\mu \|u^{k+1}(t)\|_{H_{s,\delta,A^0(u^k(t))}}^2 + 1 \right).$$

So Gronwall's inequality yields

$$\|u^{k+1}(t)\|_{H_{s,\delta,A^0(u^k(t))}}^2 \leq e^{C\mu t} \left[\|u^{k+1}(0)\|_{H_{s,\delta,A^0(u^k(0))}}^2 + Ct \right]. \quad (5.71)$$

Take now arbitrary $\epsilon > 0$, since $u^k(t) \rightarrow u(t)$ uniformly in $[0, T^{**}]$, we see from the inner-product (5.7) that there is k_0 such that

$$\|v(t)\|_{H_{s,\delta,A^0(u(t))}} \leq (1 + \epsilon) \|v(t)\|_{H_{s,\delta,A^0(u^k(t))}} \quad \text{for } k \geq k_0. \quad (5.72)$$

Combing (5.70), (5.71), (5.72) and (5.37) with the fact that $u^k(t) \rightarrow u(t)$ uniformly in $[0, T^{**}]$, we obtain

$$\begin{aligned} \limsup_{t \downarrow 0} \|u(t)\|_{H_{s,\delta,A^0(u(t))}}^2 &= \limsup_{t \downarrow 0} \|u(t)\|_{H_{s,\delta,A^0(u(t))}}^2 \\ &\leq \limsup_{t \downarrow 0} \left(\liminf_k \|u^{k+1}(t)\|_{H_{s,\delta,A^0(u(t))}}^2 \right) \\ &\leq \limsup_{t \downarrow 0} \left(\liminf_k (1 + \epsilon)^2 \|u^{k+1}(t)\|_{H_{s,\delta,A^0(u^k(t))}}^2 \right) \\ &\leq \limsup_{t \downarrow 0} \left(\liminf_k e^{C\mu t} \left[(1 + \epsilon)^2 \|u^{k+1}(0)\|_{H_{s,\delta,A^0(u^k(0))}}^2 + Ct \right] \right) \\ &= \limsup_{t \downarrow 0} \left(e^{C\mu t} \left[(1 + \epsilon)^2 \|u_0\|_{H_{s,\delta,A^0(u(0))}}^2 + Ct \right] \right) \\ &= (1 + \epsilon)^2 \|u_0\|_{H_{s,\delta,A^0(u(0))}}^2 \end{aligned}$$

which proves (5.69).

It remains to show that $\lim_{t \rightarrow t_0} (\|\partial_t u(t) - \partial_t u(t_0)\|_{H_{s-1,\delta+1}}) = 0$. Now,

$$\partial_t u = (A^0(u; t, x))^{-1} \left\{ \sum_{a=1}^3 A^a(u; t, x) \partial_a u + B(u; t, x) u + F(u; t, x) \right\}. \quad (5.73)$$

By the first step of the proof, $\|\partial_a u(t) - \partial_a u(t_0)\|_{H_{s-1,\delta+1}} \rightarrow 0$ and $\|u(t) - u(t_0)\|_{H_{s,\delta}} \rightarrow 0$. At this stage we apply Corollary 8.13 to the right hand of (5.73) and this completes the proof of Theorem 5.17. ■

5.8 Local existence for the evolution equations of Einstein-Euler system

In the previous subsections we have established the well-posedness of the first order symmetric hyperbolic system in the $H_{s,\delta}$ spaces. We would like to apply this result to the coupled system (2.26) and (2.9).

The unknowns of the evolution equations are the gravitational field $g_{\alpha\beta}$ and its first order partial derivatives $\partial_a g_{\gamma\delta}$, the Makino variable w and the velocity vector u^α . We represent them by the vector

$$U = (g_{\alpha\beta} - \eta_{\alpha\beta}, \partial_a g_{\gamma\delta}, \partial_0 g_{\gamma\delta}, w, u^\alpha, u^0 - 1), \quad (5.74)$$

here $\eta_{\alpha\beta}$ denotes the Minkowski metric. The initial data for equation (2.26) are

$$g_{ab}|_M = h_{ab}, \quad g_{0b}|_M = 0, \quad g_{00}|_M = -1, \quad -\frac{1}{2}\partial_0 g_{ab}|_M = K_{ab}.$$

where (h_{ab}, K_{ab}) are given by (3.4), and (3.5) for equation (2.9). Hence $g_{ab}|_M - \eta_{ab} = h_{ab} - I \in H_{s,\delta}$ and $(w|_M, u^a|_M, u^0|_M - 1) \in H_{s-1,\delta+2}$. Therefore we conclude that

$$U(0, \cdot) \in H_{s,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}. \quad (5.75)$$

In this situation we cannot apply directly Theorem 5.1. The idea to overcome this obstacle is the following. We first introduce some more convenience notations: $\mathbf{g} = g_{\alpha\beta} - \eta_{\alpha,\beta}$, $\partial\mathbf{g} = \partial_\alpha g_{\gamma\delta}$ (that is, $\partial\mathbf{g}$ is the set of all first order partial derivatives), $\mathbf{v} = (w, u^a, u^0 - 1)$ and $U = (\mathbf{g}, \partial\mathbf{g}, \mathbf{v})$. Since $H_{s,\delta} \subset H_{s-1,\delta}$, it follows from (5.75) that

$$U(0, \cdot) \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}. \quad (5.76)$$

If we prove the existence of $U(t, x)$ which is a solution to the coupled systems (2.26) and (2.9) with initial data in the form of (5.76) and such that $U(t, \cdot) \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$ and it is continuous with respect to this norm, then from inequality

$$\|\mathbf{g}\|_{H_{s,\delta}} \lesssim (\|\mathbf{g}\|_{H_{s-1,\delta}} + \|\partial\mathbf{g}\|_{H_{s-1,\delta+1}}), \quad (5.77)$$

we will get that $U(t, \cdot) \in H_{s,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$ and it will be continuous with respect to the norm of $H_{s,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$. Note that (5.77) certainly holds for the integral representation of the norm (7.3), and by Theorem 7.2 it holds also for the $H_{s,\delta}$ norm.

In order to achieve this we carefully examine the structure of the coupled systems (2.26) and (2.9). According to Conclusion 2.4, we can write Einstein-Euler system in the form:

$$A^0(U)\partial_t U = \sum_{a=1}^3 A^a(U)\partial_a U + B(U)U, \quad (5.78)$$

where A^α and B are 55×55 matrices such that

$$A^\alpha = \left(\begin{array}{c|c|c} I_{10} & \mathbf{0}_{10 \times 40} & \mathbf{0}_{10 \times 5} \\ \hline \mathbf{0}_{40 \times 10} & \widetilde{A}^\alpha(\mathbf{g}) & \mathbf{0}_{40 \times 5} \\ \hline \mathbf{0}_{5 \times 10} & \mathbf{0}_{5 \times 40} & \widehat{A}^\alpha(\mathbf{g}, \partial\mathbf{g}, \mathbf{v}) \end{array} \right) \quad (5.79)$$

and

$$B = \left(\begin{array}{c|c|c} \mathbf{0}_{10} & \mathbf{b}_{10 \times 40} & \mathbf{0}_{10 \times 5} \\ \hline & \widetilde{B}(\mathbf{g}, \partial\mathbf{g}, \mathbf{v}) & \\ \hline \mathbf{0}_{5 \times 10} & \mathbf{0}_{5 \times 40} & \mathbf{0}_{5 \times 5} \end{array} \right). \quad (5.80)$$

Here $\widetilde{A}^\alpha(\mathbf{g})$ is 40×40 matrix which represents system (2.26), $\widehat{A}^\alpha(\mathbf{g}, \partial\mathbf{g}, \mathbf{v})$ is 5×5 matrix of system (2.9). Both of them are symmetric and both $\widehat{A}^0(\mathbf{g})$ and $\widehat{A}^0(\mathbf{g}, \partial\mathbf{g}, \mathbf{v})$ are positive definite matrices; $\widetilde{B}(\mathbf{g}, \partial\mathbf{g}, \mathbf{v})$ is 40×55 matrix, and $\mathbf{b}_{10 \times 40}$ is a constant matrix.

A natural norm of $U = (\mathbf{g}, \partial\mathbf{g}, \mathbf{v})$ on the product space $H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$ is

$$|||U|||_{H_{s-1,\delta}}^2 = \|\mathbf{g}\|_{H_{s-1,\delta}}^2 + \|\partial\mathbf{g}\|_{H_{s-1,\delta+1}}^2 + \|\mathbf{v}\|_{H_{s-1,\delta+2}}^2. \quad (5.81)$$

Note that from Proposition (8.8) and Theorem 8.11 we have that $A^\alpha U, BU \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$, whenever $U \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$.

We formulate an inner-product in accordance with the norm (5.81) and the structure of A^0 . Let $U_1 = (\mathbf{g}_1, \partial\mathbf{g}_1, \mathbf{v}_1)$ and $U_2 = (\mathbf{g}_2, \partial\mathbf{g}_2, \mathbf{v}_2)$, similarly to (5.7) we set

$$\begin{aligned} & \langle U_1, U_2 \rangle_{s-1,\delta,A^0} \\ &:= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \int [\Lambda^{s-1}((\psi_j^2 \mathbf{g}_1)_{(2j)})]^T [\Lambda^{s-1}((\psi_j^2 \mathbf{g}_2)_{(2j)})] dx \\ &+ \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta+1)2j} [\Lambda^{s-1}((\psi_j^2 \partial\mathbf{g}_1)_{(2j)})]^T (\widetilde{A}^0)_{(2j)} [\Lambda^{s-1}((\psi_j^2 \partial\mathbf{g}_2)_{(2j)})] dx \\ &+ \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta+2)2j} [\Lambda^{s-1}((\psi_j^2 \mathbf{v}_1)_{(2j)})]^T (\widehat{A}^0)_{(2j)} [\Lambda^{s-1}((\psi_j^2 \mathbf{v}_2)_{(2j)})] dx \end{aligned} \quad (5.82)$$

and $|||U|||_{H_{s-1,\delta,A^0}}^2 = \langle U, U \rangle_{s-1,\delta,A^0}$. Since A^0 is positive definite, $|||U|||_{H_{s-1,\delta,A^0}} \sim |||U|||_{H_{s-1,\delta}}$

We can now repeat all the arguments and estimations of subsections 5.2-5.7, which are applied term-wise to the norm (5.81) and inner-product (5.82), and in this way we extend Theorem 5.1 to the product space:

Theorem 5.18 (Well posedness of hyperbolic systems in product spaces) *Let $s-1 > \frac{5}{2}$, $\delta \geq -\frac{3}{2}$ and assume the coefficient of (5.78) are of the form (5.79) and (5.80). If $U_0 \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$ and satisfies*

$$\frac{1}{\mu} U_0^T U_0 \leq U_0^T A^0 U_0 \leq \mu U_0^T U_0, \quad \mu \in \mathbb{R}^+ \quad (5.83)$$

then there exists a positive T which depends on $|||U_0|||_{H_{s-1,\delta}}$ and a unique $U(t, x)$ a solution to (5.78) such that $U(0, x) = U_0(x)$ and in addition it satisfies

$$U \in C([0, T], H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}) \cap C^1([0, T], H_{s-2,\delta+1} \times H_{s-2,\delta+2} \times H_{s-2,\delta+3}). \quad (5.84)$$

Corollary 5.19 (*Solution to the gravitational field and the fluid*) Let $\frac{7}{2} < s < \frac{2}{\gamma-1} + \frac{3}{2}$ and $\delta > -\frac{3}{2}$. Then there exists a positive T , a unique gravitational field $g_{\alpha\beta}$ solution to (2.26) and a unique (w, u^α) solution to Euler equation (2.9) such that

$$g_{\alpha\beta} - \eta_{\alpha\beta} \in C([0, T], H_{s,\delta}) \cap C^1([0, T], H_{s-1,\delta+1}) \quad (5.85)$$

and

$$(w, u^\alpha, u^0 - 1) \in C([0, T], H_{s-1,\delta+2}) \cap C^1([0, T], H_{s-2,\delta+3}). \quad (5.86)$$

Proof (of Corollary 5.19) Theorem 3.4 implies that the initial data for $g_{\alpha\beta}$ belong to $H_{s,\delta}$ and initial data for (w, u^α) are in $H_{s-1,\delta+2}$. Thus $U(0, \cdot) \in H_{s-1,\delta} \times H_{s-1,\delta+1} \times H_{s-1,\delta+2}$, where U is given by (5.74). In addition, the continuity of A^0 implies that the vector $U(0, \cdot)$ satisfies (5.83). Therefore Theorem 5.18 with inequality (5.77) give the desired result. ■

6 Quasi Linear Elliptic Equations in $H_{s,\delta}$

In this section we will establish the elliptic theory in $H_{s,\delta}$ which is essential for the solution of the constraint equations. We will extend earlier results in weighted Sobolev spaces of integer order which were obtained by Cantor [7], Choquet-Bruhat and Christodoulou [11] and Christodoulou and O’Murchadha [14] to the fractional ordered spaces. The essential tool is the a priori estimate (6.18) and proving it requires first to establish an analogous a priori estimate in Bessel potential spaces. Our approach is based on the techniques of Pseudodifferential Operators which have symbols with limited regularity and we are adopting ideas being presented in Taylor’s books [38] and [39]. A different method was derived recently by Maxwell [30].

6.1 A priori estimates for linear elliptic systems in H^s

In this section we consider a second order homogeneous elliptic system

$$(Lu)^i = \sum_{\alpha,\beta,j} a_{ij}^{\alpha\beta}(x) \partial_\alpha \partial_\beta u^j, \quad (6.1)$$

where the indexes $i, j = 1, \dots, N$ and $\alpha, \beta = 1, 2, 3$ (since only \mathbb{R}^3 is being discussed in this paper). We will use the convention

$$Lu = A(x) D^2 u, \quad (6.2)$$

where $A(x) = a_{ij}^{\alpha\beta}(x)$, $(D^2 u)_{\alpha\beta}^j = \partial_\alpha \partial_\beta u^j$ and $(A(x) D^2 u)_i = a_{ij}^{\alpha\beta}(x) (D^2 u)_{\alpha\beta}^j$. The symbol of (6.1) is $N \times N$ matrix $A(x, \xi)$, defined for all $\xi \in \mathbb{C}^3$ as follows:

$$A(x, \xi)_{ij} := - \sum_{\alpha,\beta} a_{ij}^{\alpha\beta}(x) i\xi_\alpha i\xi_\beta. \quad (6.3)$$

The following definitions are due to Morrey [32].

Definition 6.1

1. The system (6.1) is **elliptic** provided that

$$\det(A(x, \xi)) = \det\left(\sum_{\alpha, \beta} a_{ij}^{\alpha\beta}(x) \xi_\alpha \xi_\beta\right) \neq 0, \quad \text{for all } 0 \neq \xi \in \mathbb{R}^3; \quad (6.4)$$

2. The system (6.1) is **strongly elliptic** provided that for some positive λ

$$\langle A(x, \xi) \eta, \eta \rangle = \sum_{\alpha, \beta, i, j} a_{ij}^{\alpha\beta}(x) \xi_\alpha \xi_\beta \eta^i \eta^j \geq \lambda |\xi|^2 |\eta|^2. \quad (6.5)$$

Our main task is to obtain a priori estimate in the Bessel potential spaces H^s for the operator (6.1) whose coefficients $a_{\alpha\beta}^{ij}$ belong to H^{s_2} . In case s and s_2 are integers, then one may prove (6.8) below by means of induction and the classical results of Douglas and Nirenberg [15], and Morrey [32]. We will employ techniques of Pseudodifferential calculus.

If the coefficients of the matrix A belongs to H^{s_2} , then $A(x, \xi) \in H^{s_2} S_{1,0}^2$, that is, $\|\partial_\xi^\alpha A(\cdot, \xi)\|_{H^{s_2}} \leq C_\alpha (1 + |\xi|^2)^{(2-|\alpha|)/2}$. We follow Taylor and decompose

$$A(x, \xi) = A^\#(x, \xi) + A^b(x, \xi) \quad (6.6)$$

in such way that a good parametrix can be constructed for $A^\#(x, \xi)$, while $A^b(x, \xi)$ will have order less than 2. According to Proposition 8.2 in [39], for $s_2 > \frac{3}{2}$ there is $0 < \delta < 1$ such that

$$A^\#(x, \xi) \in S_{1,\delta}^2, \quad A^b(x, \xi) \in H^{s_2} S_{1,\delta}^{2-\sigma\delta}, \quad \sigma = s_2 - \frac{3}{2}$$

where $A^\#(x, \xi) = \sum_{k=0}^\infty J_{\epsilon_k} A(x, \xi) \phi_k(\xi)$, $\epsilon_k = c 2^{-k\delta}$. Here $\{\phi_k\}$ is the Littlewood-Paley partition of unity, that is, $\phi_0 \in C_0^\infty(\mathbb{R}^3)$, $\phi_0(0) = 1$, $\phi_k(\xi) = \phi_0(2^{-k}\xi) - \phi_0(2^{-k+1}\xi)$ and $\sum_{k=0}^\infty \phi_k(\xi) = 1$. The smoothing operator J_ϵ is defined as follows:

$$J_\epsilon f(x) = \phi_0(\epsilon D) f(x) = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int \epsilon^{-3} \widehat{\phi_0}\left(\frac{y}{\epsilon}\right) f(x-y) dy,$$

where $\widehat{\phi_0}$ is the inverse Fourier transform. In order that $A^\#$ will have a good parametrix we need to verify that it is a strongly elliptic. Since the original operator is strongly elliptic,

$$\begin{aligned} & \sum_{\alpha, \beta, i, j} J_{\epsilon_k} a_{ij}^{\alpha\beta}(x) \phi_k(\xi) \xi_\alpha \xi_\beta \eta^i \eta^j \\ &= \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int \left(\sum_{\alpha, \beta, i, j} \epsilon_k^{-3} \widehat{\phi_0}\left(\frac{y}{\epsilon_k}\right) a_{ij}^{\alpha\beta}(y-x) \phi_k(\xi) \xi_\alpha \xi_\beta \eta^i \eta^j \right) dy \\ &\geq \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \lambda \phi_k(\xi) |\xi|^2 |\eta|^2 \int \epsilon_k^{-3} \widehat{\phi_0}\left(\frac{y}{\epsilon_k}\right) dy = \lambda \phi_k(\xi) |\xi|^2 |\eta|^2 \phi_0(0) \\ &= \lambda \phi_k(\xi) |\xi|^2 |\eta|^2 \end{aligned}$$

for each fixed k . Summing over the k we have,

$$\langle A^\#(x, \xi)\eta, \eta \rangle = \sum_{k=0}^{\infty} \sum_{\alpha, \beta, i, j} \left(J_{\epsilon_k} a_{ij}^{\alpha\beta} \right) (x) \phi_k(\xi) \xi_\alpha \xi_\beta \eta^i \eta^j \geq \sum_{k=0}^{\infty} \lambda \phi_k(\xi) |\xi|^2 |\eta|^2 = \lambda |\xi|^2 |\eta|^2,$$

thus (6.5) holds for $A^\#$. The last step assures that $\|A^\#(x, \xi)^{-1}\| \leq \frac{1}{\lambda|\xi|^2}$ and then it follows from the identity $\partial(A^{-1}) = A^{-1}(\partial(A))A^{-1}$ that

$$\|\partial_x^\beta \partial_\xi^\alpha (A^\#(x, \xi))^{-1}\| \leq C_{\alpha\beta} (1 + |\xi|^2)^{(-2-|\alpha|+\delta|\beta|)/2},$$

that is, $(A^\#(x, \xi))^{-1} \in S_{1,\delta}^{-2}$. Hence, the operator $A^\#(x, D)$ has a parametrix $E^\#(x, D) \in OPS_{1,\delta}^{-2}$ satisfying

$$E^\#(x, D)A^\#(x, D) = I + S, \quad (6.7)$$

where $S \in OPS^{-\infty}$ (See e. g. [38] Section 0.4).

Lemma 6.2 (*An a priori estimates in H^s*) *Let $Lu = A(x)D^2u$ be a strongly elliptic system and assume $A \in H^{s_2}$, $s_2 > \frac{3}{2}$ and $0 \leq s-2 \leq s_2$. Then there is a constant C such that*

$$\|u\|_{H^s} \leq C \{ \|Lu\|_{H^{s-2}} + \|u\|_{H^{s-2}} \}. \quad (6.8)$$

Proof (of Lemma 6.2) We decompose $A(x, D)$ as in (6.2) and let $E^\#(x, D)$ be the above parametrix, then by (6.7)

$$E^\#(x, D)A(x, D)u = u + Su + E^\#(x, D)A^b(x, D)u. \quad (6.9)$$

Since $E^\#(x, D), S : H^{s-2} \rightarrow H^s$ are bounded,

$$\|E^\#(x, D)A(x, D)u\|_{H^s} = \|E^\#(x, D)Lu\|_{H^s} \leq C\|Lu\|_{H^{s-2}} \quad (6.10)$$

and

$$\|Su\|_{H^s} \leq C\|u\|_{H^{s-2}}. \quad (6.11)$$

According to [39] Proposition 8.1, (see also [38] Proposition 2.1.J)

$$A^b(x, D) : H^{s-\sigma\delta} \rightarrow H^{s-2}.$$

Hence,

$$\|E^\#(x, D)A^b(x, D)u\|_{H^s} \leq C\|A^b(x, D)u\|_{H^{s-2}} \leq C\|u\|_{H^{s-\sigma\delta}}. \quad (6.12)$$

Using the intermediate estimate $\|u\|_{H^{s-\sigma\delta}} \leq \epsilon\|u\|_{H^s} + C(\epsilon)\|u\|_{H^{s-2}}$, and combining it with (6.9)- (6.11), we obtain the estimate (6.8). \blacksquare

6.2 A priori estimates in $H_{s,\delta}$

Our main task here is to extend the a priori estimate (6.8) to $H_{s,\delta}$ -spaces and for second order elliptic systems of the form:

$$\begin{aligned} (Lu)^i &= \sum_{\alpha,\beta,j} a_{ij}^{\alpha\beta}(x) \partial_\alpha \partial_\beta u^j + \sum_{\alpha,j} b_{ij}^\alpha(x) \partial_\alpha u^j + \sum_j c_{ij}(x) u^j \\ &= A(x) D^2 u + B(x) (Du) + C(x) u. \end{aligned} \quad (6.13)$$

Here $A(x)$ is as in the previous subsection, $B(x) = b_{ij}^\alpha(x)$, $(Du)_\alpha^j = \partial_\alpha u^j$, $(B(x)Du)_i = b_{ij}^\alpha(x) \partial_\alpha u^j$ and $C(x) = c_{ij}(x)$ is $N \times N$. We introduce the following hypotheses:

Hypotheses (H)

- (H1) $\sum a_{i,j}^{\alpha,\beta}(x) \eta^i \eta^j \xi_\alpha \xi_\beta \geq \lambda |\eta|^2 |\xi|^2$ (i.e. L is strongly elliptic);
- (H2) $(A(\cdot) - A_\infty) \in H_{s_2, \delta_2}$, $B \in H_{s_1, \delta_1}$, $C \in H_{s_0, \delta_0}$
 $s_i \geq s - 2, i = 0, 1, 2$, $s_2 > \frac{3}{2}, s_1 > \frac{1}{2}, s_0 \geq 0$ and $\delta_i > \frac{1}{2} - i$, $i = 0, 1, 2$,
the matrix A_∞ has constant coefficients and $A_\infty D^2 u$ is an elliptic system, that is,
 $\det \left(\sum (a_\infty)_{ij}^{\alpha,\beta} \xi_\alpha \xi_\beta \right) \neq 0$.

We shall first derive an a priori estimate for a second order homogeneous operator

$$L_2 u = A(x) D^2 u.$$

Lemma 6.3 (*An a priori estimate for homogeneous operator in $H_{s,\delta}$*) Assume the operator L_2 satisfies hypotheses (H) and $s \geq 2$. Then

$$\|u\|_{H_{s,\delta}} \leq C \left\{ \|L_2 u\|_{H_{s-2,\delta+2}} + \|u\|_{H_{s-2,\delta}} \right\}, \quad (6.14)$$

where the constant C depends on s, δ and $\|A - A_\infty\|_{H_{s_2, \delta_2}}$.

Proof (of Lemma 6.3) According to Corollary 7.6,

$$\sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j^4 u)_{2j} \right\|_{H^s}^2$$

is an equivalent norm in $H_{s,\delta}$. The main idea of the proof is to apply Lemma 6.2 to each term of the equivalent norm above. We use the convention (6.2) and compute

$$L_2(\psi^4 u) = \psi^4 L_2(D^2 u) + \psi A(x) R(u, \psi),$$

where

$$R(u, \psi)_{\alpha\beta} = 8\psi (D\psi)_\alpha (\psi Du)_\beta + 12 (D\psi)_\alpha (D\psi)_\beta (\psi u) + 4\psi (D^2 \psi)_{\alpha\beta} (\psi u).$$

Applying the a priori estimate (6.8), we have

$$\begin{aligned}
\|u\|_{H_{s,\delta}}^2 &\lesssim \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j^4 u)_{2j} \right\|_{H^s}^2 \\
&\lesssim \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\{ \left\| L_2 ((\psi_j^4 u)_{2j}) \right\|_{H^{s-2}}^2 + \left\| (\psi_j^4 u)_{2j} \right\|_{H^{s-2}}^2 \right\} \\
&\lesssim \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\{ 2^{4j} \left\| (\psi_j^4 L_2(u))_{2j} \right\|_{H^{s-2}}^2 + \left\| (\psi_j^4 u)_{2j} \right\|_{H^{s-2}}^2 \right\} \\
&+ \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta+2)2j} \left\| (\psi_j AR(u, \psi_j))_{2j} \right\|_{H^{s-2}}^2 \\
&\lesssim \|L_2(u)\|_{H_{s-2,\delta+2}}^2 + \|u\|_{H_{s-2,\delta}}^2 + \|AR\|_{H_{s-2,\delta+2}}^2.
\end{aligned} \tag{6.15}$$

The assumption on s_2 and δ_2 enable us to use Proposition 8.8 and get

$$\begin{aligned}
\|AR\|_{H_{s-2,\delta+2}} &\leq C \left(\|(A - A_\infty)R\|_{H_{s-2,\delta+2}} + \|A_\infty R\|_{H_{s-2,\delta+2}} \right) \\
&\leq C \left(\|(A - A_\infty)\|_{H_{s_2,\delta_2}} + \|A_\infty\| \right) \|R\|_{H_{s-2,\delta+2}}.
\end{aligned} \tag{6.16}$$

Property (7.4) of ψ_j and inequality (8.32) imply

$$\|(\psi_j R)_{2j}\|_{H^{s-2}} \leq C \left(2^{-j} \|(\psi_j Du)_{2j}\|_{H^{s-2}} + 2^{-2j} \|(\psi_j u)_{2j}\|_{H^{s-2}} \right)$$

and hence $\|R\|_{H_{s-2,\delta+2}} \leq C (\|u\|_{H_{s-1,\delta}} + \|u\|_{H_{s-2,\delta}})$. Thus, inequalities (6.15) and (6.16) yields

$$\|u\|_{H_{s,\delta}} \leq C \left\{ \|L_2 u\|_{H_{s-2,\delta+2}} + \left(\|A - A_\infty\|_{H_{s_2,\delta_2}} + 1 \right) (\|u\|_{H_{s-1,\delta}} + \|u\|_{H_{s-2,\delta}}) \right\}. \tag{6.17}$$

Invoking the intermediate estimate $\|u\|_{H_{s-1,\delta}} \leq \sqrt{2\epsilon} \|u\|_{H_{s,\delta}} + C(\epsilon) \|u\|_{H_{s-2,\delta}}$ (see Proposition 8.6) and taking ϵ so that $C \left(\|A - A_\infty\|_{H_{s_2,\delta_2}} + 1 \right) \sqrt{2\epsilon} \leq \frac{1}{2}$, we obtain from (6.17) the desired estimate (6.14). \blacksquare

Lemma 6.4 (An a priori estimate in $H_{s,\delta}$) Assume the operator L of the form (6.13) satisfies hypotheses (H) and $s \geq 2$. Then

$$\|u\|_{H_{s,\delta}} \leq C \left\{ \|Lu\|_{H_{s-2,\delta+2}} + \|u\|_{H_{s-2,\delta}} \right\}, \tag{6.18}$$

where the constant C depends on s, δ and the coefficients of L .

Proof (of Lemma 6.4) By Lemma 6.3,

$$\begin{aligned}
\|u\|_{H_{s,\delta}} &\leq C \left\{ \|L_2 u\|_{H_{s-2,\delta+2}} + \|u\|_{H_{s-2,\delta}} \right\} \\
&\leq C \left\{ \|Lu\|_{H_{s-2,\delta+2}} + \|u\|_{H_{s-2,\delta}} + \|(L - L_2)u\|_{H_{s-2,\delta+2}} \right\},
\end{aligned}$$

where $(L - L_2)u = B(x)(Du) + C(x)u$. Hypothesis (H2) together with Corollary 8.2 and Proposition 8.8 give

$$\|B(x)(Du)\|_{H_{s-2,\delta+2}} \lesssim \|B\|_{H_{s_1,\delta_1}} \|Du\|_{H_{s-2,\delta+1}} \lesssim \|B\|_{H_{s_1,\delta_1}} \|u\|_{H_{s-1,\delta}}, \quad (6.19)$$

and

$$\|C(x)u\|_{H_{s-2,\delta+2}} \lesssim \|C\|_{H_{s_0,\delta_0}} \|u\|_{H_{s-2,\delta}}.$$

Finally, we apply the intermediate estimate, Proposition 8.6, to the right hand side of (6.19) and by taking ϵ sufficiently small we obtain (6.18). \blacksquare

Lemma 6.5 (Isomorphism of an operator with constant coefficients) *Let $A_\infty u := A_\infty D^2 u$ be a homogeneous elliptic system with constant coefficients. Then for any $s \geq 2$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$, the operator $A_\infty : H_{s,\delta+2} \rightarrow H_{s-2,\delta}$ is isomorphism satisfying*

$$\|u\|_{H_{s,\delta}} \leq C \|A_\infty D^2 u\|_{H_{s-2,\delta+2}}. \quad (6.20)$$

Proof (of Lemma 6.5) Both statements are true when s is an integer and under the norm (7.1) (see e. g. [11], Theorem 5.1) and by Theorem 7.2 they hold also in the $H_{s,\delta}$ norm (3.1). For s between two integers m_0 and m_1 , we have $s = s_\theta = \theta m_0 + (1 - \theta)m_1$ and $s - 2 = s_\theta - 2 = \theta(m_0 - 2) + (1 - \theta)(m_1 - 2)$, where $0 < \theta < 1$. The interpolation Theorem 8.1 implies

$$H_{s,\delta} = [H_{m_0,\delta}, H_{m_1,\delta}]_\theta \quad \text{and} \quad H_{s-2,\delta} = [H_{m_0-2,\delta}, H_{m_1-2,\delta}]_\theta.$$

Since $A_\infty^{-1} : H_{m_i-2,\delta} \rightarrow H_{m_i,\delta+2}$, $i = 0, 1$, is continuous, it follows from interpolation theory that $A_\infty^{-1} : H_{s_\theta-2,\delta} \rightarrow H_{s_\theta,\delta+2}$ is also continuous (see e. g. [41]). Hence (6.20) holds. \blacksquare

The next lemma improves the a priori estimate (6.18).

Lemma 6.6 (Improved a priori estimate) *Let L be an elliptic operator of the form (6.13) which satisfies hypotheses (H). Assume $s \geq 2$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Then for any δ' there is a constant C such that*

$$\|u\|_{H_{s,\delta}} \leq C \left\{ \|Lu\|_{H_{s-2,\delta+2}} + \|u\|_{H_{s-1,\delta'}} \right\}. \quad (6.21)$$

The constant C depends on the H_{s_i,δ_i} -norm of the coefficients of L , s , δ and δ' .

Proof (of Lemma 6.6) Let $\chi_R \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function satisfying $\text{supp}(\chi_R) \subset \{|x| \leq 2R\}$, $\chi_R(x) = 1$ for $|x| \leq R$, $0 \leq \chi_R(x) \leq 1$ and $\|\partial^\alpha \chi_R\|_\infty \leq C_\alpha R^{-|\alpha|}$. For $u \in H_{s,\delta}$ we write

$$u = (1 - \chi_R)u + \chi_R u$$

and R will be determinate later on. We start with the estimation of $\|(1 - \chi_R)u\|_{H_{s,\delta}}$ and for that purpose we use the convention (6.2) and compute

$$\begin{aligned} A_\infty(D^2(1 - \chi_R)u) &= (1 - \chi_R)A_\infty(D^2u) - 2A_\infty(D\chi_R)(Du) - A_\infty(D^2\chi_R)u \\ &= (1 - \chi_R)(Lu) + E_1 + E_2, \end{aligned} \quad (6.22)$$

where

$$E_1 = -(1 - \chi_R) \{ (A - A_\infty) (D^2 u) + B(x)(Du) + C(x)u \}$$

and

$$E_2 = - \{ 2A_\infty (D\chi_R)(Du) + A_\infty (D^2 \chi_R)u \}$$

Applying inequality (6.20) of Lemma 6.5,

$$\begin{aligned} \|(1 - \chi_R)u\|_{H_{s,\delta}} &\leq C \|A_\infty D^2((1 - \chi_R)u)\|_{H_{s-2,\delta+2}} \\ &\leq C \{ \|(1 - \chi_R)Lu\|_{H_{s-2,\delta+2}} + \|E_1\|_{H_{s-2,\delta+2}} + \|E_2\|_{H_{s-2,\delta+2}} \}. \end{aligned} \quad (6.23)$$

Since $\|(1 - \chi_R)Lu\|_{H_{s-2,\delta+2}} \lesssim \|Lu\|_{H_{s-2,\delta+2}}$ (see Proposition 8.4), it remains to estimate $\|E_1\|_{H_{s-2,\delta+2}}$ and $\|E_2\|_{H_{s-2,\delta+2}}$. We may choose δ'_i so that $\delta_i > \delta'_i > \frac{1}{2} - i$, $i = 0, 1, 2$ and then we put $\gamma = \min_{i=0,1,2}(\delta_i - \delta'_i)$. Under these conditions the application of Corollary 8.2, Propositions 8.5 and 8.8 yield

$$\begin{aligned} \|E_1\|_{H_{s-2,\delta+2}} &\leq C \|(1 - \chi_R) \{ (A - A_\infty) (D^2 u) + B(Du) + Cu \}\|_{H_{s-2,\delta+2}} \\ &\leq C \{ \|(1 - \chi_R)(A - A_\infty)\|_{H_{s_2,\delta'_2}} \|D^2 u\|_{H_{s-2,\delta+2}} \\ &\quad + \|(1 - \chi_R)B\|_{H_{s_1,\delta'_1}} \|Du\|_{H_{s-1,\delta+1}} + \|(1 - \chi_R)C\|_{H_{s_0,\delta'_0}} \|u\|_{H_{s,\delta}} \} \\ &\leq \frac{C_1}{R^\gamma} \left(\|(A - A_\infty)\|_{H_{s_2,\delta_2}} + \|B\|_{H_{s_1,\delta_1}} + \|C\|_{H_{s_0,\delta_0}} \right) \|u\|_{H_{s,\delta}} \\ &\leq \frac{C_1 \Lambda}{R^\gamma} \|u\|_{H_{s,\delta}}, \end{aligned} \quad (6.24)$$

where $\Lambda = \left(\|A - A_\infty\|_{H_{s_2,\delta_2}} + \|B\|_{H_{s_1,\delta_2}} + \|C\|_{H_{s_0,\delta_0}} \right)$.

Next, since $D\chi_R$ has compact support, Remark 8.3 and Corollary 8.2 imply that

$$\begin{aligned} \|E_2\|_{H_{s-2,\delta+2}} &\leq C(R) \left\{ \|2A_\infty((D\chi_R)(Du))\|_{H_{s-2,\delta'+1}} + \|A_\infty((D^2 \chi_R)u)\|_{H_{s-2,\delta'}} \right\} \\ &\leq C(R) \|A_\infty\| \left\{ 2\|Du\|_{H_{s-2,\delta'+1}} + \|u\|_{H_{s-2,\delta'}} \right\} \\ &\leq C(R) \|A_\infty\| \|u\|_{H_{s-1,\delta'}}. \end{aligned} \quad (6.25)$$

We turn now to the estimation of $\|\chi_R u\|_{H_{s,\delta}}$. Noting that $(\chi_R u)$ has compact support, we have by Remark 8.3, (6.18) and Proposition 8.4 that

$$\|\chi_R u\|_{H_{s,\delta}} \leq C(R) \|\chi_R u\|_{H_{s,\delta'}} \leq C(R) \{ \|L(\chi_R u)\|_{H_{s-2,\delta'+2}} + \|u\|_{H_{s-1,\delta'}} \}. \quad (6.26)$$

Similarly to (6.22) we compute

$$L(\chi_R u) = \chi_R L(u) + 2A(D\chi_R)(Du) + A(D^2 \chi_R)u + B(D\chi_R)u. \quad (6.27)$$

We estimate each term of (6.27) separately. Once again, since $\chi_R Lu$ has compact support,

$$\|\chi_R(Lu)\|_{H_{s-2,\delta'+2}} \leq C(R) \|Lu\|_{H_{s-2,\delta+2}}. \quad (6.28)$$

Next, using the second assumption of (H) , Proposition 8.8 and the compactness of $\text{supp}(\chi_R)$ we get

$$\begin{aligned}
& \|2A(D\chi_R)(Du)\|_{H_{s-2,\delta'+2}} \\
& \leq 2\|(A - A_\infty)(D\chi_R)(Du)\|_{H_{s-2,\delta'+2}} + \|A_\infty(D\chi_R)(Du)\|_{H_{s-2,\delta'+2}} \\
& \leq C \left(\|(A - A_\infty)\|_{H_{s_2,\delta_2}} + \|A_\infty\| \right) \|(D\chi_R)(Du)\|_{H_{s-2,\delta'+2}} \\
& \leq C(R) \left(\|(A - A_\infty)\|_{H_{s_2,\delta_2}} + \|A_\infty\| \right) \|Du\|_{H_{s-2,\delta'+1}} \\
& \leq C(R) \left(\|(A - A_\infty)\|_{H_{s_2,\delta_2}} + \|A_\infty\| \right) \|u\|_{H_{s-1,\delta'}}.
\end{aligned} \tag{6.29}$$

In a similar manner we estimate the other terms and together with inequalities (6.23)–(6.26), (6.28) and (6.29) we have

$$\begin{aligned}
\|u\|_{H_{s,\delta}} & \leq \|(1 - \chi_R)u\|_{H_{s,\delta}} + \|\chi_R u\|_{H_{s,\delta}} \\
& \leq C \left\{ \|Lu\|_{H_{s-2,\delta+2}} + C_2 \|u\|_{H_{s-1,\delta'}} + \frac{C_1 \Lambda}{R^\gamma} \|u\|_{H_{s,\delta}} \right\},
\end{aligned} \tag{6.30}$$

where C_1 and C_2 depend on the norms of the coefficients of L and in addition C_2 depends in R . We now take R such that $\frac{C_1 \Lambda}{R^\gamma} \leq \frac{1}{2}$, then (6.21) follows from (6.30). \blacksquare

The next two theorems are consequence of the compact embedding, Theorem 8.14, the a priori estimate (6.21) and standard arguments of Functional Analysis.

Theorem 6.7 (Semi Fredholm) *Assume the operator L satisfies hypotheses (H) , $s \geq 2$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Then $L : H_{s,\delta} \rightarrow H_{s-2,\delta+2}$ is semi Fredholm, that is,*

- (i) $\dim(\text{Ker} L) < \infty$;
- (ii) *If L is injective, then there is a constant C such that*

$$\|u\|_{H_{s,\delta}} \leq C \|Lu\|_{H_{s-2,\delta+2}}; \tag{6.31}$$

- (iii) *L has a closed range.*

Theorem 6.8 (A homotopy argument) *Let $s \geq 2$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Assume L be an elliptic operator of the form (6.13) that fulfilled the hypotheses (H) and L_t is a continuous family of operators which satisfy hypotheses (H) for $t \in [0, 1]$, $L_1 = L$ and*

$$L_t : H_{s,\delta} \rightarrow H_{s-2,\delta+2} \text{ is injective.}$$

If

$$L_0 : H_{s,\delta} \rightarrow H_{s-2,\delta+2} \text{ is an isomorphism,}$$

then the same is true for L .

The next Lemma shows that solutions to the homogeneous system have lower growth at infinity. We follow Christodoulou and O'Murchadha's proof [14].

Lemma 6.9 (Lower growth of homogeneous solutions) *Assume L satisfies hypotheses (H), $u \in H_{s,\delta}$, $s \geq 2$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. If $Lu = 0$, then $u \in H_{s,\delta'}$ for any $-\frac{3}{2} < \delta' < -\frac{1}{2}$.*

Proof (of Lemma 6.9) We recall that if $\delta' < \delta$, then it follows from Remark 7.4 that $H_{s,\delta} \hookrightarrow H_{s,\delta'}$. Hence it suffices to show the statement for $\delta' > \delta$. The conditions on δ_i imply that we may find $\delta' > \delta$ so that $\delta_i + \delta + i > \delta' + 2 - \frac{3}{2}$. Then applying the Proposition 8.8 to

$$f := A_\infty u - Lu = (A_\infty - A(x))(D^2 u) - B(x)(Du) - C(x)u,$$

we obtain that f belongs to $H_{s-2,\delta'+2}$. Now $Lu = 0$, so $A_\infty u = f$ and since $A_\infty : H_{s,\delta'} \rightarrow H_{s-2,\delta'+2}$ is isomorphism by Lemma 6.5, we conclude that hence $u \in H_{s,\delta'}$. We now replace δ by δ' repeat the above arguments. \blacksquare

6.3 Semi Linear Elliptic Equations on Asymptotically Flat Manifolds

A Riemannian 3-manifold (M, h) is asymptotically flat (AF) if there is a compact subset K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$ and the metric h tends to the identity I at infinity. A natural definition of the last statement in our case is $h - I \in H_{s',\delta'}$. We will hence assume here that $h - I \in H_{s',\delta'}$, $s' > \frac{3}{2}$ and $\delta' > -\frac{3}{2}$.

We denote by Δ_h be the Laplace-Beltrami operator on (M, h) . In the coordinates (x^1, x^2, x^3) it takes the form

$$\Delta_h = \frac{1}{\sqrt{|h|}} \partial_j \left(\sqrt{|h|} h^{ij} \partial_i \right), \quad (6.32)$$

where $|h| = \det(h_{ij})$ and $h^{ij} = (h_{ij})^{-1}$. Inserting the identity $\partial_j |h| = |h| \text{tr}(h^{ij}(\partial_j(h_{ij})))$ into (6.32), we have

$$\Delta_h = h^{ij} \partial_j \partial_i + \partial_j (h^{ij}) \partial_i + \frac{1}{2} \text{tr}(h^{ij}(\partial_j(h_{ij}))) h^{ij} \partial_i. \quad (6.33)$$

Hence, by means of Proposition 8.8, Theorem 8.11 and Remark 8.12, the elliptic operator (6.33) satisfies hypothesis (H) of Section 6.2 provided that $s \leq s'$.

Let us introduce some more notations. We denote by $\mu_h = \sqrt{|h|} dx$ the Lebesgue measure on the manifold (M, h) , $(Du \cdot Dv)_h = h^{ij} \partial_i u \partial_j v$, and $\|Du\|_h^2 = (Du \cdot Du)_h$. Integration by parts yields

$$\begin{aligned} \int (\Delta_h u) v d\mu_h &= \int \partial_j \left(\sqrt{|h|} h^{ij} \partial_i u \right) v dx \\ &= - \int h^{ij} \partial_i u \partial_j v \sqrt{|h|} dx = - \int (Du \cdot Dv)_h d\mu_h. \end{aligned} \quad (6.34)$$

Formula (6.34) holds whenever $v \in H_0^1(\mathbb{R}^3)$, $u \in H_{s,\delta}$ and $s \geq 1$. Therefore it enables us to define weak solutions on the manifold (M, h) .

Definition 6.10 (Weak solutions) A function u in $H_{s,\delta}$ is a weak solution of

$$-\Delta_h u + c(x)u = f \in H_{s-2,\delta+2}$$

on (M, h) , if

$$\int ((Du \cdot Dv)_h + cuv) d\mu_h = \int f v d\mu_h, \quad (6.35)$$

for all $v \in H_0^1(\mathbb{R}^3)$.

Remark 6.11 In case $u, v \in H_{s,\delta}$, $s \geq 2$ and $\delta \geq -1$, then by algebra $h^{ij} \partial_i u, \sqrt{|h|} \partial_j v \in H_{s-1,0}$ (see subsection 8.1). Applying the Cauchy Schwarz inequality

$$\begin{aligned} \int |(Du \cdot Dv)_h| d\mu_h &= \int |h^{ij} \partial_i u \partial_j v| \sqrt{|h|} dx \\ &\leq \left(\int (h^{ij} \partial_i u)^2 \right)^{\frac{1}{2}} \left(\int \sqrt{|h|} (\partial_j v)^2 \right)^{\frac{1}{2}} \leq \|h^{ij} \partial_i u\|_{H_{s-1,0}} \|\sqrt{|h|} \partial_j v\|_{H_{s-1,0}}, \end{aligned}$$

we see that $h^{ij} \partial_i u \partial_j v \sqrt{|h|} \in L^1(\mathbb{R}^3)$. Similarly, the integrand of the left hand side of (6.34) belongs to $L^1(\mathbb{R}^3)$. Hence, approximating u and v by C_0^∞ functions and using Lebesgue's Dominated Convergence Theorem we have

$$\int (\Delta_h u) v d\mu_h = - \int (Du \cdot Dv)_h d\mu_h, \quad u, v \in H_{s,\delta}, \text{ whenever } s \geq 2, \text{ and } \delta \geq -1. \quad (6.36)$$

In this section we will prove existence and uniqueness for the semi-linear equation

$$-\Delta_h u = F(x, u) := \sum_{i=1}^N m_i(x) h_i(u), \quad (6.37)$$

where $m_i \in H_{s_0,\delta_0}$, $m_i(x) \geq 0$, $s_0 \geq 0$, $\delta_0 > \frac{1}{2}$ and for $u > -1$ the functions h_i are decreasing, nonnegative and smooth. These conditions ensure $F(\cdot, u)$ and $\frac{\partial F}{\partial p}(\cdot, u)$ are in $H_{s-2,\delta+2}$ whenever $u \in H_{s,\delta}$ and $s \geq 2$.

Theorem 6.12 (Existence and uniqueness) Let $h - I \in H_{s',\delta'}$, $s' > \frac{3}{2}$, $\delta' > -\frac{3}{2}$, $2 \leq s \leq s'$ and $-\frac{3}{2} < \delta < -\frac{1}{2}$. Then equation (6.37) has a unique solution u in $H_{s,\delta}$. Furthermore, $0 \leq u \leq K$ for a nonnegative constant K .

In order to show Theorem 6.12 we need the weak maximal principle:

Proposition 6.13 (Weak maximal principle) Assume $c \in H_{s'-2,\delta'+2}$ is nonnegative. If $u \in H_{s,\delta}$ satisfies

$$-\Delta_h u + cu \leq 0, \quad (6.38)$$

then $u \leq 0$.

Proof (of Proposition 6.13) For $\epsilon > 0$ we put $w = \max(u - \epsilon, 0)$. It has compact support since $\lim_{x \rightarrow \infty} u(x) = 0$. Further, $Dw = Du$ a.e. in $\{u(x) > \epsilon\}$ (see e. g. [18] or [26]). Thus, $w \in H_0^1(\mathbb{R}^3)$ and $w \geq 0$, so by (6.35)

$$0 \geq \int ((Du, Dw)_h + cuw) d\mu_h = \int_{\{u \geq \epsilon\}} (\|Du\|_h^2 + cu^2) d\mu_h.$$

Therefore $u \equiv \epsilon$ in $\{u(x) \geq \epsilon\}$. Since ϵ is arbitrary, we have $u \leq 0$. ■

Proof (of Existence) The proof will be done in several steps. We define a map $\Phi : \{H_{s,\delta} \times [0, 1], u(x) > -1\} \rightarrow H_{s-2,\delta+2}$ by

$$\Phi(u, \tau) = -\Delta_h u - \tau F(x, u),$$

let $u(\tau)$ denotes a solution of $\Phi(u, \tau) = 0$ and put $J = \{0 \leq s \leq 1 : \Phi(u(s), s) = 0\}$. We will show that J is both open and closed set. Since $0 \in J$, $J = [0, 1]$ which yields the existence result.

Step 1. *The set J is open.*

Let

$$Lw := \left(\frac{\partial \Phi}{\partial u}(u, \tau) \right) (w) = -\Delta_h w - \tau \frac{\partial F}{\partial p}(\cdot, u)w$$

and

$$L_t w = -\Delta_{\{th+(1-t)I\}} w - t\tau \frac{\partial F}{\partial p}(\cdot, u)w.$$

If $L_t w = 0$, then by Lemma (6.9) $w \in H_{s,-1}$. So we may use (6.36) and get

$$\int (L_t w) w d\mu_{\{th+(1-t)I\}} = \int \left(\|Dw\|_{\{th+(1-t)I\}}^2 - t\tau \frac{\partial F}{\partial p}(\cdot, u)w^2 \right) d\mu_{\{th+(1-t)I\}}.$$

Since $\frac{\partial F}{\partial p} \leq 0$, the above yields that $L_t w = 0$ implies $w \equiv 0$ for each $t \in [0, 1]$. In addition $L_0 = -\Delta_I = -\Delta$ is an isomorphism according to Lemma 6.5. Therefore Theorem 6.8 implies that $L_1 = L$ is an isomorphism too. Thus J is open by the Implicit Function Theorem.

Step 2. $\|u(\tau)\|_{H_{s,\delta}} \leq C$ for a constant C independent of τ .

We first establish the bound in $H_{2,\delta}$ -norm. The weak maximum principle implies $u(\tau) \geq 0$ and since $F(x, p)$ is decreasing in p ,

$$\|F(\cdot, u(\tau))\|_{H_{0,\delta+2}} \leq \|F(\cdot, 0)\|_{H_{0,\delta+2}} \leq \left(\sum_{i=1}^N h_i(0)^2 \|m_i\|_{H_{0,\delta+2}}^2 \right)^{\frac{1}{2}} := K.$$

We showed in Step 1 that $\Delta_h : H_{s,\delta} \rightarrow H_{s-2,\delta+2}$ is injective, therefore from Theorem 6.7 (ii),

$$\|u(\tau)\|_{H_{2,\delta}} \leq C \| -\Delta_h u(\tau) \|_{H_{0,\delta+2}} \leq C \|F(\cdot, 0)\|_{H_{0,\delta+2}} \leq CK. \quad (6.39)$$

Now, Theorem 8.11 implies $\|h_i(u(\tau))\|_{H_{2,\delta}} \leq C\|(u(\tau))\|_{H_{2,\delta}}$ and by Proposition 8.8, $\|F(\cdot, u(\tau))\|_{H_{2,\delta}} \leq C\|u(\tau)\|_{H_{2,\delta}}$. In order to improve (6.39), we take s'' so that $s'' - 2 \leq 2$ and $s'' \leq s$. Then we may apply again (6.31) and combine it with (6.39) and Remark 7.4, we have

$$\begin{aligned} \|u(\tau)\|_{H_{s'',\delta}} &\leq C\|-\Delta_h u(\tau)\|_{H_{s''-2,\delta+2}} \leq C\|-\Delta_h u(\tau)\|_{H_{2,\delta+2}} \\ &= C\|F(\cdot, u(\tau))\|_{H_{2,\delta}} \leq C\|u(\tau)\|_{H_{2,\delta}} \leq CK. \end{aligned} \quad (6.40)$$

We have proved the boundedness in case $s'' = s$, otherwise we can repeat the same procedure as above to improve regularity until we would reach the desired regularity. It is obvious that the bound on $\|u(\tau)\|_{H_{s,\delta}}$ does not depend on τ .

Step 3. *Lipschitz continuity with respect to τ :*

Differentiation of the equation $\Phi(u(\tau), \tau) = 0$ with respect to τ gives

$$-\Delta_h u_\tau(\tau) - \tau \frac{\partial F}{\partial p} F(x, u(\tau)) u_\tau(\tau) = F(x, u(\tau)).$$

Now $\frac{\partial F}{\partial p} F(x, p) \leq 0$, so in the same way as we did in Step 1 we obtain that the operator $L = -\Delta_h - \tau \frac{\partial F}{\partial p} F(x, u(\tau)) : H_{s,\delta} \rightarrow H_{s-2,\delta+2}$ is injective. Hence, by Theorem 6.7 (ii),

$$\|u_\tau\|_{H_{s,\delta}} \leq C\|L(u_\tau)\|_{H_{s-2,\delta+2}} = C\|F(x, u(\tau))\|_{H_{s-2,\delta+2}}. \quad (6.41)$$

Next, Step 2 implies

$$\|F(x, u(\tau))\|_{H_{s-2,\delta+2}} \leq C\|u(\tau)\|_{H_{s,\delta}} \left(\sum_{i=1}^N \|m_i\|_{H_{s_0,\delta_0}} \right) \leq C. \quad (6.42)$$

Thus, combining (6.41) with (6.42) we get

$$\|u(\tau_1) - u(\tau_2)\|_{H_{s,\delta}} \leq C|\tau_1 - \tau_2|. \quad (6.43)$$

Step 4. *The set J is closed:*

Take a sequence $\{\tau_n\} \subset J$ such that $\tau_n \rightarrow \tau_0$. By (6.43), $\{u(\tau_n)\}$ is Cauchy in $H_{s,\delta}$ and therefore it converges to $u_0 \in H_{s,\delta}$. Since the map Φ is continuous, it follows that $\Phi(u_0, \tau_0) = 0$, that is $\tau_0 \in J$. This completes the proof of the existence. \blacksquare

Proof (of Uniqueness) Assume u_1 and u_2 are solutions to (6.37). We conduct the proof by showing that $\Omega := \{x : u_1(x) > u_2(x)\}$ is an empty set. Note that Ω is open since u_1 and u_2 are continuous. Put $w = u_1 - u_2$, then $-\Delta_h w = F(x, u_1) - F(x, u_2) \leq 0$ in Ω . So $w \leq 0$ in Ω by Proposition (6.13). That obviously leads to a contradiction unless Ω is empty. \blacksquare

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Appendix

7 The construction of the Spaces $H_{s,\delta}$:

The weighted Sobolev spaces of integer order below were introduced by Cantor [6] and independently by Nirenberg and Walker [33]. Nirenberg and Walker initiate the study of elliptic operators in these spaces, while Cantor used them to solve the constraint equations on asymptotically flat manifolds. For an nonnegative integer m and a real δ we define a norm

$$(\|u\|_{m,\delta}^*)^2 = \sum_{|\alpha| \leq m} \int (\langle x \rangle^{\delta+|\alpha|} |\partial^\alpha u|)^2 dx, \quad (7.1)$$

where $\langle x \rangle = 1 + |x|$. The space $H_{s,m}$ is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm (7.1). Note that the weight varies with the derivatives.

Here we will repeat Triebel's extension of these spaces into a fractional order, [40],[41]. Let $s = m + \lambda$, where m is a nonnegative integer and $0 < \lambda < 1$. One possibility of extending the ordinary integer order Sobolev spaces is the *Lipschitz-Sobolevskij Spaces*, having a norm

$$\|u\|_{m+\lambda,2}^2 = \sum_{|\alpha| \leq m} \int |\partial^\alpha u|^2 dx + \sum_{|\alpha|=m} \int \int \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{3+\lambda^2}} dx dy. \quad (7.2)$$

Hence, a reasonable definition of *weighted fractional Sobolev norm* is a combination of the norm (7.1) with (7.2):

$$(\|u\|_{s,\delta}^*)^2 = \left\{ \begin{array}{ll} \sum_{|\alpha| \leq m} \int |\langle x \rangle^{\delta+|\alpha|} \partial^\alpha u|^2 dx, & s = m \\ \sum_{|\alpha| \leq m} \int |\langle x \rangle^{\delta+|\alpha|} \partial^\alpha u|^2 dx \\ + \sum_{|\alpha|=m} \int \int \frac{|\langle x \rangle^{m+\lambda+\delta} \partial^\alpha u(x) - \langle y \rangle^{m+\lambda+\delta} \partial^\alpha u(y)|^2}{|x-y|^{3+2\lambda}} dx dy & s = m + \lambda. \end{array} \right\}, \quad (7.3)$$

here m is a nonnegative integer and $0 < \lambda < 1$. The space $H_{s,\delta}$ is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm (7.3).

The norm (7.3) is essential for the understating of the connections between the integer and the fractional order. But it has a disadvantage, namely, the double integral makes it almost impossible to establish any property (embedding, a priori estimate, etc.) needed for PDEs. We are therefore looking for an equivalent definition of the norm (7.3).

Let $K_j = \{x : 2^{j-3} \leq |x| \leq 2^{j+2}\}$, ($j = 1, 2, \dots$) and $K_0 = \{x : |x| \leq 4\}$. Let $\{\psi_j\}_{j=0}^\infty \subset C_0^\infty(\mathbb{R}^3)$ be a sequence such that $\psi_j(x) = 1$ on K_j , $\text{supp}(\psi_j) \subset \{x : 2^{j-4} \leq |x| \leq 2^{j+3}\}$, for

$j \geq 1$, $\text{supp}(\psi_0) \subset \{x : |x| \leq 2^3\}$ and

$$|\partial^\alpha \psi_j(x)| \leq C_\alpha 2^{-|\alpha|j}, \quad (7.4)$$

where the constant C_α does not depend on j .

We define now,

$$\left(\|u\|_{s,\delta}^\star \right)^2 = \begin{cases} \sum_{j=0}^{\infty} \left(2^{\delta 2j} \|\psi_j u\|_{L^2}^2 + 2^{(\delta+m)2j} \sum_{|\alpha|=m} \|\partial^\alpha(\psi_j u)\|_{L^2}^2 \right), & s = m \\ \sum_{j=0}^{\infty} \left(2^{\delta 2j} \|\psi_j u\|_{L^2}^2 + 2^{(\delta+m)2j} \sum_{|\alpha|=m} \|\partial^\alpha(\psi_j u)\|_{L^2}^2 \right) \\ + \sum_{j=0}^{\infty} 2^{(\delta+m+\lambda)2j} \left(\sum_{|\alpha|=m} \int \int \frac{|\partial^\alpha(\psi_j u)(x) - \partial^\alpha(\psi_j u)(y)|^2}{|x-y|^{3+2\lambda}} dx dy \right), & s = m + \lambda. \end{cases} \quad (7.5)$$

Proposition 7.1 (*Equivalence of norms*) *There are two positive constants c_0 and c_1 depending only on s, δ and the constants in (7.4) such that*

$$c_0 \|u\|_{s,\delta}^\star \leq \|u\|_{s,\delta}^* \leq c_1 \|u\|_{s,\delta}^\star. \quad (7.6)$$

This equivalence was proved in [40] (see also [4]).

We express these norms in terms of Fourier transform. Let

$$\hat{u}(\xi) = \mathcal{F}(u)(\xi) = \frac{1}{(2\pi)^3} \int u(x) e^{-ix \cdot \xi} dx$$

denotes the Fourier transform, put

$$\Lambda^s u = \mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}u, \quad (7.7)$$

and let H^s denotes the Bessel Potentials space having the norm

$$\|u\|_{H^s}^2 = \|\Lambda^s u\|_{L^2}^2 = \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi. \quad (7.8)$$

We also set

$$\|u\|_{h^s}^2 = \|\mathcal{F}^{-1}(|\xi|^s \mathcal{F}u)\|_{L^2}^2 = \int (|\xi|^s |\hat{u}(\xi)|)^2 d\xi.$$

It is well known that (see e. g. [20]; p. 240-241)

$$\|u\|_{h^s}^2 \simeq \begin{cases} \sum_{|\alpha|=m} \int |\partial^\alpha u|^2 dx & s = m \\ \sum_{|\alpha|=m} \int \int \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{3+2\lambda}} dx & s = m + \lambda \end{cases} \quad (7.9)$$

and since $(1 + |\xi|^2)^s \simeq (1 + |\xi|^s)$,

$$\|u\|_{H^s}^2 \simeq (\|u\|_{L^2}^2 + \|u\|_{h^s}^2). \quad (7.10)$$

Hence, by (7.5),

$$\left(\|u\|_{s,\delta}^\star\right)^2 \simeq \sum_{j=0}^{\infty} (2^{\delta 2j} \|\psi_j u\|_{L^2}^2 + 2^{(\delta+s)2j} \|\psi_j u\|_{h^s}^2) \quad (7.11)$$

We invoke now the scaling $u_\epsilon(x) := u(\epsilon x)$ ($\epsilon > 0$), then simple calculations yields $\|u_\epsilon\|_{L^2}^2 = \epsilon^{-3} \|u\|_{L^2}^2$ and $\|u_\epsilon\|_{h^s}^2 = \epsilon^{2s-3} \|u\|_{h^s}^2$. Combining the later one with (7.10), we have

$$\|u_\epsilon\|_{H^s}^2 \simeq \epsilon^{-3} (\|u\|_{L^2}^2 + \epsilon^{2s} \|u\|_{h^s}^2). \quad (7.12)$$

Setting $\epsilon = 2^j$, multiplying (7.12) by 2^{3j} and inserting it in (7.11), we conclude

$$\left(\|u\|_{s,\delta}^\star\right)^2 \simeq \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2^j}\|_{H^s}^2. \quad (7.13)$$

The last one is the most convenience form of norm for applications and therefore the right hand side of (7.13) defines the norm of $H_{s,\delta}$ space (see Definition 3.1).

Combining Proposition 7.1 with (7.11) and (7.13) we get:

Theorem 7.2 (Equivalence of norms, Triebel) *There are two positive constant c_0 and c_1 depending only on s, δ and the constants in (7.4) such that*

$$c_0 \|u\|_{H_{s,\delta}} \leq \|u\|_{s,\delta}^* \leq c_1 \|u\|_{H_{s,\delta}}. \quad (7.14)$$

Remark 7.3 *Theorem 7.2 enables us to use both sorts of the norms (7.3) and (3.1), and for each application we will use the suitable type of norm.*

Remark 7.4 *Let $s' \leq s$ and $\delta' \leq \delta$, then the inclusion $H_{s,\delta} \hookrightarrow H_{s',\delta'}$ follows easily from the representations (7.8) and (3.1) of the norms.*

Remark 7.5 *The functions $\{\psi_j\}$ are constructed by means of a composition of exponential functions. Hence, for any positive γ there holds*

$$c_1(\gamma, \alpha) |\partial^\alpha \psi_j^\gamma(x)| \leq |\partial^\alpha \psi_j(x)| \leq c_2(\gamma, \alpha) |\partial^\alpha \psi_j^\gamma(x)|. \quad (7.15)$$

Therefore the equivalence (7.6) remains valid with ψ_j^γ replacing ψ_j and hence

$$\sum_j 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^\gamma u)_{(2^j)}\|_{H^s}^2 \simeq \left(\|u\|_{s,\delta}^\star\right)^2 \simeq \sum_j 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{(2^j)}\|_{H^s}^2. \quad (7.16)$$

Corollary 7.6 (Equivalence of norms) *For any positive γ , there are two positive constants c_0 and c_1 depending on s, δ and γ such that*

$$c_0 \|u\|_{H_{s,\delta}}^2 \leq \sum_j 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^\gamma u)_{(2^j)}\|_{H^s}^2 \leq c_1 \|u\|_{H_{s,\delta}}^2. \quad (7.17)$$

8 Some Properties of $H_{s,\delta}$

Theorem 8.1 (*Complex interpolation, Triebel*)

Let $0 < \theta < 1$, $0 \leq s_0 < s_1$ and $s_\theta = \theta s_0 + (1 - \theta)s_1$, then

$$[H_{s_0,\delta}, H_{s_1,\delta}]_\theta = H_{s_\theta,\delta}, \quad (8.1)$$

where (8.1) is a complex interpolation.

As a consequence of the interpolation Theorem 8.1 we get

Corollary 8.2 ($H_{s,\delta}$ -norm of a derivative)

$$\|\partial_i u\|_{H_{s-1,\delta+1}} \leq \|u\|_{H_{s,\delta}} \quad (8.2)$$

Proof (of Corollary 8.2) Let m be a positive integer and define $T : H_{m,\delta} \rightarrow H_{m-1,\delta+1}$ by $T(u) = \partial_i u$. Using the norm (7.1) we see that $\|T(u)\|_{H_{m-1,\delta+1}} \leq \|u\|_{H_{m,\delta}}$. So (8.2) follows from Theorem 8.1. ■

Remark 8.3 If $\text{supp } u \subset \{|x| \leq R\}$, then for any δ

$$c_1(R)\|u\|_{H^s} \leq \|u\|_{H_{s,\delta}} \leq c_2(R)\|u\|_{H^s}. \quad (8.3)$$

This follows from the integral representation of the norm (7.1) and the interpolation (8.1).

Proposition 8.4 (*Multiplication by smooth functions*)

Let $N \geq s$ be an integer. Assume $f \in C^N(\mathbb{R}^3)$ satisfies $\sup |D^k f| \leq K$ for $k = 0, 1, \dots, N$, then

$$\|fu\|_{H_{s,\delta}} \leq C_s K \|u\|_{H_{s,\delta}}. \quad (8.4)$$

Proof (of Proposition 8.4) By the well known property of H^s

$$\|fu\|_{H^s} \leq C_s K \|u\|_{H^s} \quad (8.5)$$

we have

$$\begin{aligned} \|fu\|_{H_{s,\delta}}^2 &= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j fu)_{2j}\|_{H^s}^2 \\ &\leq (C_s K)^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2j}\|_{H^s}^2 = (C_s K)^2 \|u\|_{H_{s,\delta}}^2. \end{aligned} \quad (8.6)$$

■

Proposition 8.5 (Multiplication by cutoff function) *Let $\chi_R \in C^\infty(\mathbb{R}^3)$ satisfies $\chi_R(x) = 1$ for $|x| \leq R$, $\chi_R(x) = 0$ for $|x| \geq 2R$ and $|\partial^\alpha \chi_R| \leq c_\alpha R^{-|\alpha|}$. Then for $\delta' < \delta$ there holds*

$$\|(1 - \chi_R)u\|_{H_{s,\delta'}} \leq \frac{C(\delta, \delta')}{R^{\delta-\delta'}} \|u\|_{H_{s,\delta}}. \quad (8.7)$$

Proof (of Proposition 8.5) Let J_0 be the smallest integer such that $R \leq 2^{J_0-3}$. Then $(1 - \chi_R)\psi_j = 0$ for $j = 0, 1, \dots, J_0 - 1$. Hence

$$\begin{aligned} \|(1 - \chi_R)u\|_{H_{s,\delta'}}^2 &= \sum_{j=J_0}^{\infty} 2^{(\frac{3}{2}+\delta')2j} \|(\psi_j(1 - \chi_R)u)_{2j}\|_{H^s}^2 \\ &\leq C^2 \sum_{j=J_0}^{\infty} 2^{(\frac{3}{2}+\delta')2j} \|(\psi_j u)_{2j}\|_{H^s}^2 = C^2 \sum_{j=J_0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} 2^{(\delta'-\delta)2j} \|(\psi_j u)_{2j}\|_{H^s}^2 \\ &\leq C^2 2^{(\delta'-\delta)2J_0} \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2j}\|_{H^s}^2 \leq \frac{C^2}{(8R)^{(\delta-\delta')2}} \|u\|_{H_{s,\delta}}^2. \end{aligned} \quad (8.8)$$

■

Proposition 8.6 (An intermediate estimate) *Let $0 \leq s_0 < s < s_1$ and $\varepsilon > 0$, then there is a constant $C = C(\varepsilon)$ such that*

$$\|u\|_{H_{s,\delta}} \leq \sqrt{2\varepsilon} \|u\|_{H_{s_1,\delta}} + C \|u\|_{H_{s_0,\delta}}, \quad (8.9)$$

holds for all $u \in H_{s_1,\delta}$.

Proof (of Proposition 8.6) Inequality (8.9) is well known in H^s spaces. We apply it to each term of the norm (3.1) and get

$$\begin{aligned} \|u\|_{H_{s,\delta}}^2 &= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2j}\|_{H^s}^2 \\ &\leq 2\varepsilon^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2j}\|_{H^{s_1}}^2 + 2C^2(\varepsilon) \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2j}\|_{H^{s_0}}^2 \\ &= 2\varepsilon^2 \|u\|_{H_{s_1,\delta}}^2 + 2C^2(\varepsilon) \|u\|_{H_{s_0,\delta}}^2, \end{aligned}$$

■

Proposition 8.7 (An intermediate estimate) *Let $0 < s' < s$, then*

$$\|u\|_{H_{s',\delta}} \leq \|u\|_{H_{s,\delta}}^{\frac{s'}{s}} \|u\|_{H_{0,\delta}}^{1-\frac{s'}{s}}. \quad (8.10)$$

Proof (of Proposition 8.7) Using Hölder inequality we get $\|u\|_{H^{s'}} \leq \|u\|_{H^s}^{\frac{s'}{s}} \|u\|_{L^2}^{1-\frac{s'}{s}}$ and applying it and using again Hölder inequality yields

$$\begin{aligned}
\|u\|_{H_{s',\delta}}^2 &= \sum_j 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2j}\|_{H^{s'}}^2 \\
&\leq \sum_j 2^{(\frac{3}{2}+\delta)2j(\frac{s'}{s})} \|(\psi_j u)_{2j}\|_{H^s}^{2\frac{s'}{s}} 2^{(\frac{3}{2}+\delta)2j(\frac{s-s'}{s})} \|(\psi_j u)_{2j}\|_{L^2}^{2\frac{s-s'}{s}} \\
&\leq \left(\sum_j 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2j}\|_{H^s}^2 \right)^{\frac{s'}{s}} \left(\sum_j 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2j}\|_{L^2}^2 \right)^{\frac{s-s'}{s}} \\
&= (\|u\|_{H_{s,\delta}})^{\frac{2s'}{s}} (\|u\|_{H_{0,\delta}})^{\frac{2(s'-1)}{s}}.
\end{aligned}$$

■

8.1 Algebra

Proposition 8.8 (Algebra in $H_{s,\delta}$)

If $s_1, s_2 \geq s$, $s_1 + s_2 > s + \frac{3}{2}$ and $\delta_1 + \delta_2 \geq \delta - \frac{3}{2}$, then

$$\|uv\|_{H_{s,\delta}} \leq C \|u\|_{H_{s_1,\delta_1}} \|v\|_{H_{s_2,\delta_2}}. \quad (8.11)$$

Proof (of Proposition 8.8) By Corollary 7.6,

$$\|uv\|_{H_{s,\delta}}^2 \simeq \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^2 uv)_{2j}\|_{H^s}^2. \quad (8.12)$$

We apply the classic algebra property $\|uv\|_{H^s} \leq C \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}$ (see e. g. [38] Ch. 3, Section 5), to each term of the norm (8.12) and then we use Cauchy Schwarz inequality,

$$\begin{aligned}
\|uv\|_{H_{s,\delta}}^2 &\leq C \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^2 uv)_{2j}\|_{H^s}^2 \\
&\leq C^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{2j}\|_{H^{s_1}}^2 \|(\psi_j v)_{2j}\|_{H^{s_2}}^2 \\
&\leq C^2 \sum_{j=0}^{\infty} \left(2^{(\frac{3}{2}+\delta_1)2j} \|(\psi_j u)_{2j}\|_{H^{s_1}}^2 \right) \left(2^{(\frac{3}{2}+\delta_2)2j} \|(\psi_j v)_{2j}\|_{H^{s_2}}^2 \right) \\
&\leq C^2 \left(\sum_{j=0}^{\infty} \left(2^{(\frac{3}{2}+\delta_1)2j} \|(\psi_j u)_{2j}\|_{H^{s_1}}^2 \right)^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} \left(2^{(\frac{3}{2}+\delta_2)2j} \|(\psi_j v)_{2j}\|_{H^{s_2}}^2 \right)^2 \right)^{\frac{1}{2}} \\
&\leq C^2 \left(\sum_{j=0}^{\infty} \left(2^{(\frac{3}{2}+\delta_1)2j} \|(\psi_j u)_{2j}\|_{H^{s_1}}^2 \right) \right) \left(\sum_{j=0}^{\infty} \left(2^{(\frac{3}{2}+\delta_2)2j} \|(\psi_j v)_{2j}\|_{H^{s_2}}^2 \right) \right) \\
&\leq C^2 \|u\|_{H_{s_1,\delta_1}}^2 \|v\|_{H_{s_2,\delta_2}}^2.
\end{aligned}$$

■

8.2 Fractional power $|u|^\gamma$

In [23] Kateb showed that if $u \in H^s \cap L^\infty$, $1 < \gamma$ and $0 < s < \gamma + \frac{1}{2}$, then

$$\||u|^\gamma\|_{H^s} \leq C(\|u\|_{L^\infty})\|u\|_{H^s}. \quad (8.13)$$

Proposition 8.9 (Fractional power in $H_{s,\delta}$) Let $u \in H_{s,\delta} \cap L^\infty$, $1 < \gamma$, $0 < s < \gamma + \frac{1}{2}$ and $\delta \in \mathbb{R}$, then

$$\||u|^\gamma\|_{H_{s,\delta}} \leq C(\|u\|_{L^\infty})\|u\|_{H_{s,\delta}}. \quad (8.14)$$

Proof (of Proposition 8.9) Inequality (8.14) is a direct consequence of the Corollary 7.6 and (8.13). Because

$$\begin{aligned} \||u|^\gamma\|_{H_{s,\delta}}^2 &\simeq \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j^\gamma |u|^\gamma)_{(2j)}\|_{H^s}^2 \\ &\leq (C(\|u\|_{L^\infty}))^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j u)_{(2j)}\|_{H^s}^2 \leq (C(\|u\|_{L^\infty}))^2 \|u\|_{H_{s,\delta}}^2. \end{aligned} \quad (8.15)$$

■

8.3 Moser type estimates

Y. Meyer proved the below Moser type estimate [31]. See also Taylor [38].

Theorem 8.10 (Third Moser inequality for Bessel potentials spaces)

Let $F : \mathbb{R}^m \rightarrow \mathbb{R}^l$ be C^{N+1} function such that $F(0) = 0$. Let $s > 0$ and $u \in H^s \cap L^\infty$. Then

$$\|F(u)\|_{H^s} \leq K\|u\|_{H^s}, \quad (8.16)$$

where

$$K = K_N(F, \|u\|_{L^\infty}) \leq C\|F\|_{C^{N+1}} (1 + \|u\|_{L^\infty}^N), \quad (8.17)$$

here N is a positive integer such that $N \geq [s] + 1$.

We generalize this important inequality to the $H_{s,\delta}$ spaces.

Theorem 8.11 (Third Moser inequality in $H_{s,\delta}$)

Let $F : \mathbb{R}^m \rightarrow \mathbb{R}^l$ be C^{N+1} function such that $F(0) = 0$. Let $s > 0$, $\delta \in \mathbb{R}$ and $u \in H_{s,\delta} \cap L^\infty$. Then

$$\|F(u)\|_{H_{s,\delta}} \leq K\|u\|_{H_{s,\delta}}, \quad (8.18)$$

The constant K in (8.18) depends on one in (8.17) and in addition on δ .

Proof (of Theorem 8.11) Let $\{\psi_j\}$ be the sequence satisfying (7.4) and $\Psi_j(x) = \frac{1}{\varphi(x)}\psi_j(x)$, where $\varphi(x) = \sum_{j=0}^{\infty} \psi_j(x)$. From the properties of the sequence $\{\psi_j\}$, it follows that $1 \leq \varphi(x) \leq 7$. So the sequence $\{\Psi_j\} \subset C_0^\infty(\mathbb{R}^3)$ and $\sum_{j=0}^{\infty} \Psi_j(x) = 1$. From (7.12) we conclude that

$$\|u_\epsilon\|_{H^s}^2 \leq C \max\{\epsilon^{2s-3}, \epsilon^{-3}\} \|u\|_{H^s}^2 \quad (8.19)$$

and with the combination of (5.17) and Meyer's Theorem 8.10 we have,

$$\begin{aligned} \|F(u)\|_{H_{s,\delta}}^2 &= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j(F(u)))_{(2^j)}\|_{H^s}^2 \\ &= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| \left(\psi_j F \left(\sum_{k=0}^{\infty} \Psi_k(x) u \right) \right)_{(2^j)} \right\|_{H^s}^2 \\ &= \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| \left(\psi_j F \left(\sum_{k=j-4}^{j+3} \Psi_k(x) u \right) \right)_{(2^j)} \right\|_{H^s}^2 \\ &\leq CK^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \sum_{k=j-4}^{j+3} \|(\Psi_k u)_{(2^j)}\|_{H^s}^2 \\ &\leq CK^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \sum_{k=j-4}^{j+3} \|((\Psi_k u)_{2^{j-k}})_{(2^k)}\|_{H^s}^2 \quad (8.20) \\ &\leq CK^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \sum_{k=j-4}^{j+3} \max\{2^{(2s-3)(j-k)}, 2^{-3(j-k)}\} \|(\Psi_k u)_{(2^k)}\|_{H^s}^2 \\ &\leq C(s)K^2 \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \sum_{k=j-4}^{j+3} \|(\psi_k u)_{(2^k)}\|_{H^s}^2 \\ &\leq C(s, \delta)K^2 \sum_{j=0}^{\infty} \sum_{k=j-4}^{j+3} 2^{(\frac{3}{2}+\delta)2k} \|(\psi_k u)_{(2^k)}\|_{H^s}^2 \\ &\leq 7C(s, \delta)K^2 \sum_{k=0}^{\infty} 2^{(\frac{3}{2}+\delta)2k} \|(\psi_k u)_{(2^k)}\|_{H^s}^2 \leq 7C(s, \delta)K^2 \|u\|_{H_{s,\delta}}^2. \end{aligned}$$

■

Remark 8.12 If $F(0) \neq 0$ and $F(0) \in H_{s,\delta}$, then we can apply Theorem 8.11 to $\tilde{F}(u) := F(u) - F(0)$ and get

$$\|F(u)\|_{H_{s,\delta}} \leq \|\tilde{F}(u)\|_{H_{s,\delta}} + \|F(0)\|_{H_{s,\delta}} \leq K\|u\|_{H_{s,\delta}} + \|F(0)\|_{H_{s,\delta}}. \quad (8.21)$$

We may apply Theorem 8.11 to the estimate the difference $F(u) - F(v)$.

Corollary 8.13 (A difference estimate in $H_{s,\delta}$) Suppose F is a C^{N+2} function and $u, v \in H_{s,\delta} \cap L^\infty$. Then

$$\|F(u) - F(v)\|_{H_{s,\delta}} \leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty}) (\|u\|_{H_{s,\delta}} + \|v\|_{H_{s,\delta}}) \|u - v\|_{H_{s,\delta}}. \quad (8.22)$$

Proof (of Corollary 8.13) Put $\tilde{F}(u) = F(u) - F(0) - DF'(0)u$, then it suffices to show inequality (8.22) for \tilde{F} . Now,

$$\tilde{F}(u) - \tilde{F}(v) = \int_0^1 \left(D\tilde{F}(tu + (1-t)v) \right) (u - v) dt = G(u, v)(u - v), \quad (8.23)$$

where $G(u, v) = \int_0^1 D\tilde{F}(tu + (1-t)v) dt$. Since $G(0, 0) = \int_0^1 D\tilde{F}(0) dt = 0$, we can apply Theorem 8.11 to $G(u, v)$ and get:

$$\|G(u, v)\|_{H_{s,\delta}} \leq C(\|u\|_{L^\infty}, \|v\|_{L^\infty}) (\|u\|_{H_{s,\delta}} + \|v\|_{H_{s,\delta}}). \quad (8.24)$$

Applying Proposition (8.8) to the right side of (8.23), we have

$$\left\| \tilde{F}(u) - \tilde{F}(v) \right\|_{H_{s,\delta}} \leq C \|G(u, v)\|_{H_{s,\delta}} \|u - v\|_{H_{s,\delta}} \quad (8.25)$$

and its combination with (8.24) gives (8.22). ■

8.4 Compact embedding

Theorem 8.14 (Compact embedding) Let $0 \leq s' < s$ and $\delta' < \delta$, then the embedding

$$H_{s,\delta} \hookrightarrow H_{s',\delta'}. \quad (8.26)$$

is compact.

Proof (of Theorem 8.14) Let $\{u_n\} \subset H_{s,\delta}$ be a sequence with $\|u_n\|_{H_{s,\delta}} \leq 1$. Since $H_{s,\delta}$ is a Hilbert space there is a subsequence, denoted by $\{u_n\}$, which converges weakly to u_0 . We will complete the proof by showing that $u_n \rightarrow u_0$ strongly in $H_{s',\delta'}$.

Let $\chi_R \in C_0^\infty$ such that $\chi_R(x) = 1$ for $|x| \leq R$ and $\text{supp}(\chi_R) \subset B_{2R}$. For a given $\epsilon > 0$, we take R such that $2 \frac{C(\delta,\delta')}{R^{\delta-\delta'}} < \epsilon$, where $C(\delta,\delta')$ is the constant of inequality (8.7) of Proposition 8.5. For a bounded domain Ω , it is known that the embedding $H^s(\Omega) \hookrightarrow H^{s'}(\Omega)$ is compact and from Remark 8.3 it follows that $\|\chi_R u_n\|_{H^s} \leq C$, where C does not depend on n . Hence $\chi_R u_n$ converges strongly to \hat{u}_0 in $H^{s'}$. In addition, we have that $\chi_R u_n \rightarrow \chi_R u_0$ weakly in H^s and hence $\chi_R u_n \rightarrow \chi_R u_0$ weakly in $H^{s'}$. Thus the sequence $\{\chi_R u_n\}$ converges both strongly to \hat{u}_0 and weakly to $\chi_R u_0$ in $H^{s'}$, hence $\hat{u}_0 = \chi_R u_0$ (because $\lim_n (\chi_R u_n - \chi_R u_0), (\hat{u}_0 -$

$\chi_R u_0\rangle_{s'} = \langle (\hat{u}_0 - \chi_R u_0), (\hat{u}_0 - \chi_R u_0) \rangle_{s'} = \|\hat{u}_0 - \chi_R u_0\|_{H^{s'}}^2 = 0$ $\lim_n \|\chi_R u_n - \chi_R u_0\|_{H^{s'}, \delta'} = 0$, hence we may take n sufficiently large so that $\|\chi_R u_n - \chi_R u_0\|_{H^{s'}, \delta'} < \epsilon$. Therefore

$$\begin{aligned} \|u_n - u_0\|_{H^{s'}, \delta'} &= \|(\chi_R u_n - \chi_R u_0) + (1 - \chi_R)(u_n - u_0)\|_{H^{s'}, \delta'} \\ &\leq \|(\chi_R u_n - \chi_R u_0)\|_{H^{s'}, \delta'} + \|(1 - \chi_R)(u_n - u_0)\|_{H^{s'}, \delta'} \\ &< \epsilon + \frac{C}{R^{\delta - \delta'}} \|(u_n - u_0)\|_{H_{s, \delta}} \leq \epsilon + \frac{C}{R^{\delta - \delta'}} (\|u_n\|_{H_{s, \delta}} + \|u_0\|_{H_{s, \delta}}) \\ &\leq \epsilon + 2 \frac{C(\delta, \delta')}{R^{\delta - \delta'}} < 2\epsilon \end{aligned} \quad (8.27)$$

and that completes the proof. \blacksquare

8.5 Embedding into the continuous

We introduce the following notations. For a nonnegative integer m and $\beta \in \mathbb{R}$, we set

$$\|u\|_{C_\beta^m} = \sum_{|\alpha| \leq m} \sup_x ((1 + |x|)^{\beta + |\alpha|} |\partial^\alpha u(x)|)$$

Let C_β^m be the functions space corresponding to the above norms.

Theorem 8.15 (*Embedding into the continuous*)

If $s > \frac{3}{2} + m$ and $\delta + \frac{3}{2} \geq \beta$, then any $u \in H_{s, \delta}$ has a representative $\tilde{u} \in C_\beta^m$ satisfying

$$\|\tilde{u}\|_{C_\beta^m} \leq C \|u\|_{H_{s, \delta}}. \quad (8.28)$$

Proof (of Theorem 8.15) We first show (8.28) when $m = 0$. In order to make notations simpler we will use the convention $2^k = 0$ if $k < 0$. Recall that $\psi_j(x) = 1$ on $K_j := \{2^{j-3} \leq |x| \leq 2^{j+2}\}$. Using the known embedding $\sup_x |u(x)| \leq C \|u\|_{H^s}$ (see e. g. [27]), we have

$$\begin{aligned} \sup_x (1 + |x|)^\beta |u(x)| &\leq 2^\beta \sup_{j \geq -1} \left(2^{\beta j} \sup_{\{2^j \leq |x| \leq 2^{j+1}\}} |u(x)| \right) \\ &\leq 2^\beta \sup_{j \geq -1} (2^{\beta j} \sup |\psi_j(x) u(x)|) = 2^\beta \sup_{j \geq -1} (2^{\beta j} \sup |\psi_j(2^j x) u(2^j x)|) \\ &\leq 2^\beta C \sup_{j \geq -1} (2^{\beta j} \|(\psi_j u)_{2^j}\|_{H^s}) \leq 2^\beta C \sup_{j \geq -1} (2^{(\frac{3}{2} + \delta)j} \|(\psi_j u)_{2^j}\|_{H^s}) \leq 2^\beta C \|u\|_{H_{s, \delta}}. \end{aligned} \quad (8.29)$$

If $m > 1$, $s > \frac{3}{2} + m$ and $\delta + \frac{3}{2} \geq \beta$, then $\partial^\alpha u \in H_{s - |\alpha|, \delta + |\alpha|}$ for $1 \leq |\alpha| \leq m$. So we may apply (8.29) to $\partial^\alpha u$ and obtain $\|\partial^\alpha u\|_{C_{\beta + k}^m} \leq C \|\partial^\alpha u\|_{H_{s - |\alpha|, \delta + |\alpha|}}$. \blacksquare

8.6 Density

Theorem 8.16 (*Density of C_0^∞ functions*)

- (a) The class $C_0^\infty(\mathbb{R}^3)$ is dense in $H_{s,\delta}$.
- (b) Given $u \in H_{s,\delta}$ and $s' > s \geq 0$. Then for $\rho > 0$ there is $u_\rho \in C_0^\infty(\mathbb{R}^3)$ and a positive constant $C(\rho)$ such that

$$\|u_\rho - u\|_{H_{s,\delta}} \leq \rho \quad \text{and} \quad \|u_\rho\|_{H_{s',\delta}} \leq C(\rho)\|u\|_{H_{s,\delta}}. \quad (8.30)$$

Property (a) was proved by Triebel [41]. We prove both of them here since (b) relies on (a).

Proof (of Theorem 8.16) Let J_ϵ be the standard mollifier, that is, $\text{supp}(J_\epsilon) \subset B(0, \epsilon)$, $\hat{J}_\epsilon(\xi) = \hat{J}_1(\epsilon\xi) = \hat{J}(\epsilon\xi)$ and $\hat{J}(0) = 1$. It is well known that for any $v \in H^s$, $\|J_\epsilon * v - v\|_{H^s} \rightarrow 0$ and that $J_\epsilon * v$ belongs to $C^\infty(\mathbb{R}^3)$. In addition, we claim that there is $C = C(\epsilon, s, s')$ such that

$$\|J_\epsilon * v\|_{H^{s'}} \leq C\|v\|_{H^s}. \quad (8.31)$$

Indeed, since $J \in C_0^\infty(\mathbb{R}^3)$, $|\hat{J}(\xi)| \leq C_m(1+|\xi|)^{-m}$ for any integer m . Therefore, for a given s' and ϵ , we chose m and the constant $C(\epsilon, s, s')$ so that $(1+|\xi|^2)^{s'-s}|\hat{J}(\epsilon\xi)|^2 \leq C^2(\epsilon, s, s')$. Hence

$$\begin{aligned} \|J_\epsilon * v\|_{H^{s'}}^2 &= \int (1+|\xi|^2)^{s'} |\hat{J}(\epsilon\xi)|^2 |\hat{v}(\xi)|^2 d\xi = \int (1+|\xi|^2)^s |\hat{v}(\xi)|^2 (1+|\xi|^2)^{s'-s} |\hat{J}(\epsilon\xi)|^2 d\xi \\ &\leq C^2(\epsilon, s, s') \int (1+|\xi|^2)^s |\hat{v}(\xi)|^2 d\xi = C^2(\epsilon, s, s') \|v\|_{H^s}^2. \end{aligned}$$

- (a) Given $u \in H_{s,\delta}$ and $\rho > 0$ we may chose N such that

$$\sum_{j=N-2}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \|(\psi_j(u))_{(2j)}\|_{H^s}^2 \leq \rho^2.$$

Set now $u_N = \sum_{j=0}^N \Psi_k u$, where Ψ_k is defined as in the proof of Theorem 8.11. We use and get

$$\begin{aligned} \|fu\|_{H^s} &\leq C_s K \|u\|_{H^s} \quad (8.32) \\ \|u - u_N\|_{H_{s,\delta}}^2 &\leq \sum_{j=0}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| \left(\psi_j \left(\sum_{k=N+1}^{\infty} \Psi_k u \right) \right)_{(2j)} \right\|_{H^s}^2 \\ &= \sum_{j=N-2}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| \left(\sum_{k=j-3}^{j+4} \psi_j \Psi_k u \right)_{(2j)} \right\|_{H^s}^2 \leq C \sum_{j=N-2}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \sum_{k=j-3}^{j+4} \left\| (\psi_j u)_{(2j)} \right\|_{H^s}^2 \\ &\leq 7C \sum_{j=N-2}^{\infty} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j u)_{(2j)} \right\|_{H^s}^2 = 7C\rho^2. \end{aligned}$$

Now u_N has compact support, therefore $J_\epsilon * u_N \in C_0^\infty(\mathbb{R}^3)$ and

$$\|J_\epsilon * u_N - u_N\|_{H_{s,\delta}}^2 \leq \sum_{j=0}^{N+4} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j(J_\epsilon * u_N))_{(2^j)} \right\|_{H^s}^2 \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

- (b) Let $u \in H_{s,\delta}$ and $\rho > 0$, then by (a) we can chose N sufficiently large and ϵ small so that $\|J_\epsilon * u_N - u\|_{H_{s,\delta}} < \rho$ and by (8.31)

$$\begin{aligned} \|J_\epsilon * u_N\|_{H_{s',\delta}}^2 &\leq \sum_{j=0}^{N+4} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j(J_\epsilon * u_N))_{(2^j)} \right\|_{H^{s'}}^2 \\ &\leq C^2(\epsilon, s, s') \sum_{j=0}^{N+4} 2^{(\frac{3}{2}+\delta)2j} \left\| (\psi_j u_N)_{(2^j)} \right\|_{H^s}^2 \leq C^2 C^2(\epsilon, s, s') \|u\|_{H_{s,\delta}}^2. \end{aligned}$$

Thus, $u_\rho = J_\epsilon * u_N$.

■

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